Proof theoretical strength of Martin-Löf Type Theory with W-type and one universe

Dissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
der Fakultät für Mathematik
der Ludwig-Maximilians-Universität
München

vorgelegt von
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eingereicht am 10. September 1993

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Tag der mündlichen Prüfung:

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Chapter 1

Introduction

In this thesis, we bring together two different fields of Mathematical Logic, Martin-Löf's type theory and proof theory. Since we hope, that some type theorists will read it as well, which do not know proof theory very well, we will take the chance to try to motivate the work of proof theory.

Proof theoretical strength is a measure for theories. First results go back to Gentzen ([Gen36], [Gen38] and [Gen43]) how showed the consistency of Peano Arithmetic PA by means of transfinite induction up to ϵ_0. He used finitistic arguments. Schütte has given a clearer version of it by using the ω-rule. We want to sketch this argument very briefly.

This was done by interpreting the proof of PA in a semi-formal, non finitistic, system, where we have the ω-rule for $\forall$-introduction:

$$\frac{A(n) \quad \text{for all } n \in \mathbb{N}}{\forall x.A(x)}$$

If we have interpreted the base case and induction step of an induction axiom in this system, with conclusion $\forall x.A(x)$, we can prove $A(n)$ by using the base case and n-times the induction step, and conclude $\forall x.A(x)$ in our non finitistic system. Note, that the height of the nth premise is the sum of the height of the base case and n-times the height of the induction step, so the height of the new derivation is infinite. For the interpretation, we needed the cut-rule

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}$$

(here Γ is a sequence of formulas, which can be read as the disjunction of the formulas). This rule corresponds fo a lemma: If we have proven A and a lemma, that proves, that from A follows B, (classically this is $\neg A \vee B$, written as a sequence $B, \neg A$), we can conclude B. In our interpretation of the induction, in the n-th premise we have used n cuts, to conclude, that from $A(0)$ and $A(Sm), \neg A(m)$ follows $A(n)$. We can eliminate these cuts, by induction over the derivation. A cut-free proof is a truth-definition, so, there is no cut-free proof of $0 = 1$ and we have shown that $PA \not\vdash 0 = 1$.

To do this, we needed induction over trees. We can measure the size of the trees by ordinals and replace the induction over trees by transfinite induction. A careful analysis shows, that induction up to ϵ_0 is sufficient. We can even formalize this analysis in PA plus transfinite induction up to ϵ_0. So PA cannot prove transfinite induction up to ϵ_O, since by Gödel, PA does not prove its own consistency. For each $\alpha < \epsilon_0$ we can prove transfinite induction up to α in PA. So we have proven, that ϵ_0 is the supremum of ordinals, up to which we can prove transfinite induction in PA, and we say, the proof theoretical strength of PA is ϵ_0, $|PA| = \epsilon_0$.

Now we have found a measure for the strength of a theory. The theorem, that PA proves transfinite induction up to ω_n, where $\omega_0 = 0$, $\omega_{n+1} = \omega^{\omega_n}$, $\epsilon_0 = sup\{\omega_n | n \in \mathbb{N}\}$, are in a logical sense the most complicated sentences, which we can prove in PA – the diagonalization of it can not be proven any more in PA.

This sort of analysis or related work could be carried out for many other theories as well (see for instance the books [BFPS81] [BS88], [Gir87], [Poh89], [Sch77a], [Tak87], the overviews [Poh91], [Poh92],[Rat91b], [WW92]), it has turned out to be a good measure, and gives an ordering of theories. The landscape below $|\Delta^2_1 - CA + BI|$ is now very well known (see for instance Rathjen's dissertation [Rat89]), knowing the proof theoretical strength gives a very good intuition, what can essentially be done in a theory. For instance, Γ_0 is the bound for predicative theories – for the analysis of stronger theories we need totally new methods, where we can prove complete cut elimination with ordinal analysis only for a restricted set of formulas (in subsystems of analysis for Π^0_2-sentences). $|\Pi^1_2 - CA|$, which is not calculated yet, and we do not know if it can ever be calculated, seems to be the bound, after which some sort of hyper-impredicativity starts – when we come closer to $\Pi^1_2 - CA$, the ordinals explode.

We think, that this analysis is important, For instance, if we are playing around with Martin-Löf's type theory, we can find out, how different concepts change the strength of it.

Let us therefore have a look on the landscape of Martin-Löf's type theory, seeing, what is known about it. We follow the introduction of [Pal92]. Aczel has shown in [Acz77] that Martin-Löf's type theory with one universe but no W-type has strength $|\widehat{ID}_1| = \phi_{\epsilon_0}0$. Martin-Löf's type theory with n Universes has strength $|\widehat{ID}_n| = \alpha_n$ with $\alpha_0 := \epsilon_0$, $\alpha_{n+1} := \phi_{\alpha_n}0$, (see in [Fef82b]), and if we have infinitely many universes, but still no W-type, the theory $(ML_{<\omega})$ has strength $|\widehat{ID}_{<\omega}| = \Gamma_0$, the last result due to Beeson and Feferman (see remark II.6(iv) in [Fef82b] or [Bee85]). The extensional version, $(ML^e_{<\omega})$ has the same strength (see Beeson, [Bee85]). This is astonishing, since these are all predicative bounds. As soon as we add the W-type, the proof theoretical ordinals become impredicative. Palmgren has shown in [Pal92] (part of his thesis [Pal91]) the following results: If we have type theory with one universe and an induction principle on it, we have $|ML^i_1| \geq |ID_1|$. Adding extensionality gives the same strength, and if we have infinitely many universes with induction principles, we have $|ML^e_{<\omega}V| \geq |ML_{<\omega}V| \geq |ID_{<\omega}|$. For intensional type theory with one universe and W-type, he has shown $|ML^e_1 W| \geq |ML_1 W| \geq |(\Delta^1_2 - CA)|$, but conjectured, that the strength will certainly far exceed this bound.

The strength we prove, is in fact far bigger, it is slightly bigger than the strength of $|KPi| = |(\Delta^1_2 - CA) + (BI)|$. Until recently (work of Rathjen, [Rat90], [Rat91a], [Rat92], see also [Buc93], [Sch91a], [Sch91b], [Sch92b]), KPi has been essentially the strongest theory, for which proof theoretical analysis could be carried out. For the author, $|KPi|$ is an ordinal which seems to be an ordinal of significance similar to that of Γ_0. It is quite difficult, really to exceed this strength, and most of mathematics can be carried out within theories of this strength. Feferman stated in remark I, 13.2 and 4 of his paper [Fef75], that Martin-Löf's type theory with one Universe and W-type, as well as his theory T_0, which is of strength KPi, are in accordance with Bishop's approach (see [BB85]) — all the constructive mathematics can be carried out in these theories. So the result proven here underlines this arguments, and shows, that Martin-Löf's type theory is important for the foundations of Mathematics. (For more discussions on the constructivism as an approach towards better foundations of mathematics, see [Bee85], [DT88], [Fef79], [Fef82a]and for discussions on intuitionism [Tro73]) It also justifies the use of it as basic theory for proof development systems.

Precisely we calculate the proof theoretical strength of intensional Russel-, extensional Tarski- and extensional Russel-version of Martin-Löf's type theory with one Universe and the W-type, $ML_1^iW_R$, $ML_1^eW_T$ and $ML_1^eW_R$, namely $|ML_1^iW_R| = |ML_1^eW_T| = |ML_1^eW_R| = |\bigcup_{n\in\mathbb{N}} KPi_n^+| = \psi_{\Omega_1}\Omega_{I+\omega}$. Here Russel-version stands for the formulation à là Russel of the universe, Tarski-version for the formulation à là Tarski, described in [ML84].

The thesis is organized as follows: In the first part, we introduce Martin-Löf's type theory and Kripke-Platek set theory. In chapter 2 we give an introduction on Martin-Löf's type theory. To make a precise definition of the substitution, we introduce sets of r-objects, g-terms, g-types, r-terms and r-types, which should contain all the terms and types occurring in Martin-Löf's type theory (g-terms and g-types correspond to the Tarski-formalization, r-terms and r-types to the Russel-formalization). These concepts will be needed afterwards for the interpretation of $ML_1^eW_T$ in KPi_n^+. We define then the theories $ML_1^iW_T$, $ML_1^eW_T$, $ML_1^iW_R$, $ML_1^eW_R$. In chapter 3 we compare the Tarski and the Russel-versions and in chapter 4 we introduce the Kripke-Platek set theories, we need.

In the following second big part, we prove $|ML_1^eW_T| \le \psi_{\Omega_1}\Omega_{I+\omega}$ by interpreting $ML_1^eW_T$ in set theoretical systems of Kripke-Platek style, called KPi^+ and KPi_n^+. This embedding is a quite general and flexible method, which can be adopted to variations of Martin-Löf's type theory. In chapter 5, we develop, how to interpret terms and types in KPi^+. Types will be introduced as sets of pairs of terms, which are considered to be equal in that type. In this chapter, we really see, how W-type and the Universe extend the proof theoretical strength of Martin-Löf's type theory.

In chapter 6 we first prove essentially, that the interpretation is correct, that is

$$\text{if } ML_1^eW_T \vdash a : A \text{ then } KPi^+ \vdash pair(a, a) \in A^*$$

Further we conclude, that the extended version $|ML_1^eW_{T,U}|$ can be interpreted as well.

In chapter 7 we interpret sentences of Arithmetic A as $\widehat{A}$ in $ML_1^eW_T$ and prove, that for arithmetical sentences A from $ML_1^eW_T \vdash a \in \widehat{A}$ follows $KPi^+ \vdash A$. It follows, $|ML_1^eW_T| \le |KPi^+|$. Since every proof can be interpreted actually in KPi_n^+ for some n, and in the essential lemmas KPi^+ can be replaced by KPi_n^+, we conclude, that

$$ML_1^eW_T \vdash r \in \widehat{A} \Rightarrow \exists n.KPi_n^+ \vdash A.$$

In the third part we show, that $ML_1^iW_R$ proves the well-ordering up to $\psi_{\Omega_1}\Omega_{I+n}$ for each $n < \omega$, again using very flexible methods. Since $ML_1^iW_R$ is a subsystem of $ML_1^eW_R$ follows that the upper bound is a lower bound as well.

In chapter 8 we actually carry out the well-ordering proof. The hard work is, having introduced a concept for the power set of $\mathbb{N}$, to define $W(X)$, some sort of very strong inductive definition on ordinals. Having introduced these two concepts, we can use well known techniques for carrying out such strong proofs in systems of analysis. We will refer to a version, ([Buc90]) where we need only some properties of an ordinal denotation system, such as the Bachmann property, which makes relatively clear, what is going on, and avoids, having to deal too much in this chapter with ordinals.

The properties of this denotation system, which can all be proven by using only induction on N, are verified in the very technical chapter 9. This chapter is only important to check, that the properties are valid, the proofs are not necessary for understanding, what is going on in Martin-Löf's type theory.

In chapter 10 we compare the system introduced in chapter 9 with the system [Buc92b]. The first one we used for the well ordering proof, in the latter we expressed the upper bound. We show, that lower bound and upper bound are the same. This chapter rewards the hard work of chapter 9, to do all in a system containing the Veblen-function. Without this function, the work would have been far easier, but it is quite complicated (see for instance my Diplomarbeit [Set90] or now the new work [Sch92a]) to express the Veblen-functions within a system, that does not contain them.

Future research: We did not prove, but hope to do it soon, that all arithmetical Π^0_2-sentences provable in KPi^+_n can be proven in $ML^i_1W_R$ as well. There are some ideas, using the well-ordering proof to do this. From there might follow, too, some bounds for the fast or slow growing hierarchy (see for instance proof theoretical work on this in [Sch77b], [Sch92c], [BW87], [Wai89], [Ara91], [Buc91b], [Buc91a])

Remark: While the author was finishing his thesis, M. Rathjen has told him, that in collaboration with E. Griffor independently and in parallel he has found similar results, although he has not submitted it for publication, yet, or told anything to the author.

Acknowledgement: The main amount of the research for this thesis was done as part of the EC-Twinning-Project Leeds-Munich-Oslo on “Proof theory and computation”, where most of part one was written during a visit of the author at the University of Leeds. It benefited very much from the collaboration of the three universities, since it is an example for the application of proof theoretical techniques, which have a long tradition in Munich, on type theoretical systems, on which semantical investigations are carried out in Oslo, by using the experience in the relationship between proof theory and recursion theory of Leeds. The author wants to thank D. Norman, H. Schwichtenberg for help and inspiration, S. Wainer for motivating me very much and especially W. Buchholz, who gave me an excellent insight in proof theory.

Part I

Definition of the Theories

Chapter 2

Definition of the formal system of Martin-Löf's type theories

In this chapter we define the Martin-Löf's type theories, we are going to analyze afterwards. After an introduction to it, we define the formal language (definition 2.1). Although this contradicts a little bit the spirit of Martin-Löf's type theory, we introduce a set of b-objects, containing all terms and types that will actually occur in Martin-Löf's type theory. This approach makes it easier to define substitution, allows to make definitions for all b-objects and will later be used for the interpretation of Martin-Löf's type theory in KPi_n^+. The b-objects a very general class of objects containing all terms and types, b-substitutions, and the concept of substitution (we introduced b-objects for making the definition of substitution easy). Afterwards we define (definition 2.3) g-terms, g-types, g-context-pieces, g-contexts, g-substitutions, g-judgements and g-statements, which contain all the terms, types etc. which occur in the Tarski-version of Martin-Löf's type theory, and the corresponding r-terms, r-types (definition 2.4), terms and types occurring in the Russel-version. Next we define, what derivability in the four theories $ML_1^iW_T$, $ML_1^iW_R$, $ML_1^eW_T$ and $ML_1^eW_R$ (definition 2.5) and give a few remarks.

Martin-Löf's type theory is a system used to formalize constructive mathematics, as used informally in [BB85]. One of the key ideas was, that, constructively seen, proofs and programs are the same, as are propositions and program specifications. In Martin-Löf's type theory we have only one concept for propositions, program specifications and sets, the types. The advantage of this concept is, that it makes clear, why the Curry-Howard terms for realization of a proposition $A \to B$ and a proposition $\forall x \in N.B$ are both λ-terms: a realization of $A \to B$ maps any element of the proposition A to an element of the proposition B, and realization of $\forall x \in N.B$ is a function mapping elements n of the set N to elements of the proposition $B(n)$ — they can both be represented in Martin-Löf's type theory by the same concept $\Pi x \in A.B$: elements of this type are functions, mapping elements r of the type A to elements of the type $B(r)$.

$A(x)$ depends on the choice of x, therefore we need in Martin-Löf's type theory types, depending on variables. This concept makes obvious, why, if we apply a Curry-Howard term for the formula $\forall x \in A$ to an element r of type $\mathbb{N}$, we get a Curry-Howard term of the formula $A[x/r]$, a formula, which depends on the actual choice of r. The rules are designed in such a way, that this is naturally explained.

Martin-Löf's type theory differs from other mathematical theories in the sense that it has no fixed set of terms and types, but they are introduced during the derivation process. Therefore we have additional judgements of the form A *type* and in the rules we need premises which guarantee, that the type-part of the conclusion is a type. For instance to conclude that $i(a) : A+B$ (where $A+B$ is the disjunctive union of the types A and B or, considered as a proposition, $A+B$ stands for $A \vee B$) the premise $a : A$ is not sufficient, we need additionally a premise $B :$ type.

Terms, considered as programs, can be evaluated. If we reduce a term, we want to get an equivalent term, equivalent within the type, therefore we need an equality on the types.

In the formalization of [DT88] statements of Martin-Löf's type theory are of the form $\Gamma \Rightarrow \Theta$ where Γ is a context and Θ is a judgement. A context is a sequence of assumptions $x_0 : A_0, \ldots, x_n : A_n$, where x_i are free variables, A_i are types with free variables among $\{x_0, \ldots, x_{i-1}\}$. Judgements are of the form A *type*, $t : A$, $t = s : A$ (t are equal elements of the type A) and $A = B$ (A and B are equal types).

We have four sorts of rules: the first sort is the sort of type introduction rules for introducing types (in the extensional case we conclude here that certain types are equal, in the intensional version we conclude statements "$A :$ type"). The second sort is the sort of introduction rules for introducing the canonical elements of a type, the elements formed by a introductory constructor of the type. Next sort is the sort of elimination rules, rules for deriving from some elements of one type elements of another type,(for instance the induction or primitive recursion for the N type, the induction or recursion over trees for the W-type, application for the Π-type). Last sort is the sort of equality rules, which are rules for conversion of a term, we get by introducing and immediately eliminating an element of a type. For propositions, this last sort corresponds to cut elimination, for sets it corresponds to the evaluation of a term. Additionally we have some general rules, some sort of structural rules.

The types, we have in this version are the following:

The type N, corresponding to the natural numbers.

the type N_k, corresponding to a set with k elements, where N_0 is the empty type or the falsum for the propositions, having as elimination rule the ex-falsum-quodlibet, N_1 corresponds to the always true formula, N_2 is the type of booleans. The types N_k, $k > 0$ have case distinction as elimination rule.

the type $\Pi x \in A.B$, which corresponds to the formula $\forall x \in A.B(x)$ or considered as a set to the set of functions mapping $x : A$ to an element of $B(x)$.

the type $\Sigma x \in A.B$, corresponding to the formula $\exists x \in A.B(x)$, if $x \notin FV(B)$ to the formula $A \wedge B$ or as a set to the set of pairs $\{pair(x, y) | x : A, y : B(x)\}$.

the type $A + B$, which corresponds to the formula $A \vee B$, and as a set to the disjoint union of A and B.

the type $I(A, r, s)$ which corresponds to the formula $r = s$ for $r, s : A$.

the type $Wx \in A.B$, which corresponds to a set of trees with branching degrees $B(x)$ for $x : A$.

Universe U, which is a type, closed under all other operations that create types.

For a more general introduction to Martin-Löf's type theory the reader might refer to Martin-Löf's fascinating monograph [ML84], or the books [Bee85] and [DT88].

There are two versions of Martin-Löf's type theory, an extensional and an intensional version. The extensional version has additional rules for deriving equalities between types and between terms. For the formulation of the universe, there are two versions: the Russel-version (we write $ML_1^eW_R$ for the extensional and $ML_1^iW_R$ for the intensional Russel-version; in the Russel-version the elements of the Universe are types) and the Tarski-version (we write $ML_1^eW_T$ for the extensional and $ML_1^iW_T$ for the intensional Tarksi-version; here the elements of the Universe are indices for types, and, if we have

$\Gamma \Rightarrow a : U$, we can conclude $\Gamma \Rightarrow T(a)$ type). For some remarks on these two versions, see [ML84].

For the formulation of the version à là Russel, we follow essentially [DT88] (an earlier version of this is [Tro87]) in the version à là Tarski, we remain in the spirit of [DT88], taking rules as defined in [Smi84]. From [DT88], we only differ a little bit in the exact notation, and the following aspects:

We add the type N_k, as in [ML84], which can be derived from the other types, but does not cause any problems, see [DT88].

We make α-conversion explicit (see the rule $(ALPHA)$, note, that from this rule follows full α-conversion).

We add some rules to $(REFL)$, which are shown in [DT88] to be derived rules, but only for a weaker system. We just do not want to bother about deriving again these rules, since they do not at all cause any difficulties.

At the end of this introduction, having praised a lot Martin-Löf's type theory, the author wants to say a few critical remarks to it. One problem is, that the set of rules is immense, and this makes it difficult to handle it proof theoretically. It seems to be a good system from the foundational point of few, but it is hard to be analyzed and might be difficult for the practical use, although various proof assistants are based on it. Another problem is, that substitution is not very well solved. For this proof theoretical analysis it was necessary, to introduce the concept of b-objects, a set theoretical concept, which is not in the spirit of Martin-Löf's type theory. Additionally, this caused a lot of problems in the interpretation, which just reflects, the unsatisfactory solution of this detail.

Definition 2.1 *We define the formal language of Martin-Löf's type theory L_{ML}.*

We assume some infinite set of variables $Var_{ML} = \{z_i^{ML} | i \in I\!N\}$, where $i \neq j \rightarrow z_i^{ML} \neq z_j^{ML}$.

Further we have the basic symbols
type, $\Rightarrow$, :, ,, (,), =, $\in$.

Constructor symbols are either term constructors or type constructors, , defined as follows:

We have the 0-ary term constructors 0, $\underline{r}$, $\underline{n}$, n_k (for each $n < k$, $n, k \in I\!N$ a new symbol n_k), and $\underline{n}_k$ (for each $k \in I\!N$ a new symbol $\underline{n}_k$);
the 1-ary term constructors: S, i, j, p_0, p_1;
the 2-ary term constructors: p, sup, R, Ap, $\widetilde{+}$, π, σ, w;
the 3-ary term constructors: D, P, $\widetilde{i}$;
for every $n \in I\!N$ an $n+1$-ary term constructor C_n.
the 0-ary type constructors N_k (for each $k \in I\!N$), N, U;
the 1-ary type constructor T;
the 2-ary type constructors: $+$, Π, Σ, W;
the 3-ary type constructor: I;

Definition 2.2 *(a) To shorten the definition of free Variables and substitution we define inductively a set of b-objects (basic objects), which will contain all terms and types occurring in Martin-Löf's type theory, together with the set of free Variables $FV(t)$ for every b-object t .*

(BO1) *If $x \in Var_{ML}$, then x is a b-object, $FV(x) := \emptyset$.*
(BO2) *If C is a n-ary constructor symbol, $t_1, \ldots, t_n$ are b-objects, then*
$C(t_1 \cdots t_n)$ is a b-object,

$FV(C(t_1 \cdots t_n)) := FV(t_1) \cup \cdots \cup FV(t_n)$.

(BO3) *If* $x \in Var_{ML}$, t *a b-object, then* $(\lambda x.t)$ *is a b-object,* $FV((\lambda x.t)) := FV(t) \setminus \{x\}$.

The b-terms are defined as the b-objects, except that rule (BO2) is only applied for C *being a* n*-ary term-constructor.*

We write $+, \tilde{+}$ *infix (that is* $(a+b)$ *for* $+(a,b)$*)* $(x)t$ *for* $\lambda x.t$, $(x,y)t$ *for* $\lambda x.\lambda y.t$, $(x,y,z)t$ *for* $\lambda x.\lambda y.\lambda z.t$. *Further, if* $S \in \{\Sigma, \Pi, W, \sigma, \pi, w\}$, $Sx \in s.t := S(s,(x)t)$.
We omit brackets, as long as there is no confusion, using the convention, that the scope of $\lambda x.$ *is as long as possible, for instance* $\lambda x.st$ *should be read as* $\lambda x.(st)$*, and* $s+t+v := s+(t+v)$ *similarly for longer sums and for* $\tilde{+}$ *(and later for other defined operators like* $\rightarrow$, $\wedge$*).*

(b) *A b-substitution is a sequence* $[x_1/t_1, \ldots, x_n/t_n]$, , *where* $x_i \in Var_{ML}$, t_i *are b-objects. (*$x_i = x_j$ *is allowed, and in this situation, only the first occurrence of a variable will be relevant)*
If $\vec{x} = x_1, \ldots, x_n$, $\vec{t} = t_1, \ldots, t_n$, *then* $[\vec{x}/\vec{t}] := [x_1/t_1, \ldots, x_n/t_n]$.

If $X \subset Var_{ML}$, $\{i_j | j \leq m\} = \{i | x_i \notin X\}$, $i_0 < i_1 <, \ldots, i_m$, *then* $[x_1/t_1, \ldots, x_n/t_n] \setminus X := [x_{i_1}/t_{i_1}, \ldots, x_{i_n}/t_{i_n}]$. *Similar, we define* $[\vec{x}/\vec{t}] \cap X := [\vec{x}/\vec{t}] \setminus (Var_{ML} \setminus X)$.

(c) *We define for a b-substitution* $[x_1/t_1, \ldots, x_n/t_n]$ *and a b-term* s *the application of the substitution to the term, written* $s[x_1/t_1, \ldots, x_n/t_n]$*, having as result a b-term, (we define* $s[\vec{x}/\vec{t}] := s[x_1/t_1, \ldots, x_n/t_n]$*), and the relation "*$s[x_1/t_1, \ldots, x_n/t_n]$ *is an allowed substitution" (or* $s[\vec{x}/\vec{t}]$ *is an allowed substitution).*

The definition is by induction on the definition of the b-term s.

Case (B1): If $x \in Var_{ML}$, *then*
$x[x_1/t_1, \ldots, x_n/t_n] := \begin{cases} x & \text{if } x_i \neq x \text{ for all } i = 1, \ldots, n, \\ t_i & \text{if } x_i = x, \ (i \text{ minimal}) \end{cases}$,
and $x[x_1/t_1, \ldots, x_n/t_n]$ *is always an allowed substitution.*

Case (B2): If C *is a* m*-ary constructor symbol,* $s_1, \ldots, s_m$ *are b-objects, then*
$C(s_1 \cdots s_m)[x_1/t_1, \ldots, x_n/t_n] := C(s_1[x_1/t_1, \ldots, x_n/t_n], \ldots, s_m[x_1/t_1, \ldots, x_n/t_n])$,
$C(s_1, \ldots, s_m)[x_1/t_1, \ldots, x_n/t_n]$ *is an allowed substitution iff for all* $i = 1, \ldots, m$,
$s_i[x_1/t_1, \ldots, x_n/t_n]$ *is an allowed substitution.*

Case (B3): If $x \in Var_{ML}$, t *is a b-object,* $[x'_1/t'_1, \ldots, x'_{n'}/t'_{n'}] := [x_1/t_1, \ldots, x_n/t_n] \setminus \{x\}$,
then $(\lambda x.t)[x_1/t_1, \ldots, x_n/t_n] := \lambda x.(t[x'_1/t'_1, \ldots, x'_{n'}/t'_{n'}])$,
and $(\lambda x.t)[x_1/t_1, \ldots, x_n/t_n]$ *is an allowed substitution,*
iff $t[x'_1/t'_1, \ldots, x'_{n'}/t'_{n'}])$ *is an allowed substitution, and for all* $i = 1 \cdots n'$, *whenever* $x'_i \in FV(t)$, *then* $x \notin FV(t'_i)$.

(d) *We define* α*-equality as the least relation between b-objects, such that for* t, t', t'', t_i, t'_i *b-objects,* $x, x' \in Var_{ML}$ *:*

$t =_\alpha t$.

If $t =_\alpha t'$, $t' =_\alpha t''$, *then* $t =_\alpha t''$.

If $t =_\alpha t'$, *then* $\lambda x.t =_\alpha \lambda x.t'$.

If $x' \notin FV(\lambda x.t)$, $t[x/x']$ *is an allowed substitution,* $\lambda x.t =_\alpha \lambda x'.(t[x/x'])$.

If $t_i =_\alpha t'_i$, C *a* n*-ary constructor, then* $C(t_1, \ldots, t_n) =_\alpha C(t'_1, \ldots, t'_n)$.

(e) *We will write* rs *for* $Ap(r,s)$ *(if* r,s *are b-objects) (we will need then parenthesis, using the usual conventions, especially, that the binding of* $\lambda x.$ *and* $\Pi x \in A.$ *etc. is as long as possible).*

$A \times B := A \wedge B := \Sigma z_i^{ML} \in A.B$, $A \to B := \Pi z_i^{ML} \in A.B$, *where* i *is minimal, such that* $z_i^{ML} \notin FV(B)$.

Definition 2.3 (a) *We define inductively a set of generalized terms, called g-terms, which will contain all terms occurring in Martin-Löf's type theory, and is a subset of the b-objects:*

If $x \in Var_{ML}$, *then* x *is a g-term.*
If $n < k$, $n,k \in I\!N$, *then* n_k *is a g-term and if* $k \in I\!N$, *then* $\underline{n}_k$ *is a g-term.*
If r,s,t *are g-terms,* $x,y,z,x' \in Var_{ML}$, $x \neq y \neq z \neq x$, *then* 0, $\underline{r}$, $\underline{n}$, Sr, $\lambda x.r$, $p(r,s)$, $sup(r,s)$, $i(r)$, $j(r)$, $P(r,s,(x,y)t)$, $Ap(r,s)$, $p_0(r)$, $p_1(r)$, $R(r,(x,y,z)s)$, $D(r,(x)s,(x')t)$, $\pi x \in r.s$, $\sigma x \in r.s$, $wx \in r.s$, $r \tilde{+} s$, $\tilde{i}(r,s,t)$ *are g-terms*
If $n \in I\!N$ *and* r, $s_1,\ldots,s_n$ *are g-terms, then* $C_n(r,s_1,\ldots,s_n)$ *is a g-term.*

(b) *We define inductively a set of generalized types, called g-types, which will contain all types occurring in Martin-Löf's type theory:*

If $k \in I\!N$, *then* N_k *is a g-type.*
N, U *are g-types.*
If A,B *are g-types,* $x \in Var_{ML}$, r,s *g-terms, then* $\Pi x \in A.B$, $\Sigma x \in A.B$, $Wx \in A.B$, $A+B$, $I(A,r,s)$, $T(r)$ *are g-types.*

(c) *The generalized context pieces (g-context-pieces) are (possibly empty) sequences* $x_1 : A_1,\ldots,x_n : A_n$, *where* $x_i \in Var_{ML}$ *are distinct, and* A_i *are g-types.* $Var(x_1 : A_1,\ldots,x_n : A_n) := \{x_1,\ldots,x_n\}$. *Note that, if* Γ, Γ' *are g-context-pieces,* $Var(\Gamma') \cap Var(\Gamma) = \emptyset$, *then* Γ,Γ' *(the concatenation of the strings* Γ, "," *and* Γ'*) is a g-context-piece.*

Further we define the set of generalized contexts (g-contexts), which is a subset of the g-context-pieces:

The empty string Γ *is a g-context.*
If Γ *is a g-context,* $x \in Var_{ML} \setminus Var(\Gamma)$, A *a g-type,* $FV(A) \subset Var(\Gamma)$, *then* $\Gamma, x : A$ *is a g-context.*

We define for a g-context-piece Γ, $x \in Var_{ML}$ *and a g-term* r *the relation "t is substitutable for* x *in* Γ*" and* $\Gamma[x/t]$ *as follows:*
If Γ *is empty, then* t *is substitutable for* x *in* Γ *and* $\Gamma[x/t] := \Gamma$.
t *is substitutable for* x *in* $\Gamma, x' : A$ *if* t *is substitutable for* x *in* Γ *and in* A *and* $(\Gamma, x' : A)[x/t] := \Gamma[x/t], x' : (A[x/t])$.

(d) *A g-substitution is a b-substitution* $[x_1/t_1,\ldots,x_n/t_n]$, *where* t_i *are g-terms.*

(e) *A generalized judgement, short g-judgement, is a string* $A : type$ *or* $A = B$ *or* $r : A$ *or* $r = s : A$, *where* A,B *are g-types, and* r,s *are g-terms. The set of free variables is* $FV(A : type) := FV(A)$, $FV(A = B) := FV(A) \cup FV(B)$, $FV(r : A) := FV(r) \cup FV(A)$ $FV(r = s : A) := FV(r) \cup FV(s) \cup FV(A)$.
We define $(A : type)[x/r] := A[x/r] : type$, $(A = B)[x/r] := A[x/r] = B[x/r]$, $(r : A)[x/t] := r[x/t] : A[x/t]$, $(r = s : A)[x/t] := r[x/t] = s[x/t] : A[x/t]$.
A generalized statement, short g-statement, is a string $\Gamma \Rightarrow \Theta$, *where* Γ *is a g-context,* Θ *is a g-judgement.*

Definition 2.4 *We define inductively the set of Russel-terms, short r-terms, and the Russel-types, short r-types, which will contain all terms and types occurring in the Russel-version of Martin-Löf's type theory, and are subsets of the b-objects:*

If $n < k$, $n, k \in I\!N$, *then* n_k *is a r-term,*
if $k \in I\!N$, *then* N_k *is a r-term,*
if r_i *are r-terms,* $n \in I\!N$ *,then* $C_n(r_0, r_1, \ldots, r_n)$ *is a r-term,*
if $r, s, t,$ *are r-terms,* $x, y, z, x' \in Var_{ML}$, $x \neq y \neq z \neq x$, *then* x, 0, $\underline{r}$, N, Sr, $\lambda x.r$, $p(r, s)$, $sup(r, s)$, $i(r)$, $j(r)$, $P(r, s, (x, y)t)$, $Ap(r, s)$, $p_0(r)$, $p_1(r)$, $R(r, (x, y, z)s)$, $D(r, (x)s, (x')t)$, $\Pi x \in r.s$, $\Sigma x \in r.s$, $Wx \in r.s$, $r + s$ *and* $I(r, s, t)$ *are r-terms.*
U *is a r-type.*
If s, t *are r-terms,* A, B *are r-types,* $x \in Var_{ML}$, *then* s, $\Pi x \in A.B$, $\Sigma x \in A.B$, $Wx \in A.B$, $A + B$ *and* $I(A, s, t)$ *are r-types.*
R-context-pieces, r-contexts, r-judgements, r-statements are defined as the corresponding g-definitions, only replacing g-terms by r-terms, g-types by r-types.

Definition 2.5 *Definition of intensional and extensional Martin-Löf's type theory with* W*-type and one Universe. We will in the following define the rules of intensional*

($ML_1^iW_T$ *is the Tarski- and* $ML_1^iW_R$ *the Russel-version) and extensional Martin-Löf's type theory (*$ML_1^eW_T$ *is the Tarski- and* $ML_1^eW_R$ *the Russel-version), which are of the form*

$$(Rule) \quad \frac{\Gamma_1 \Rightarrow \Theta_1 \cdots \Gamma_n \Rightarrow \Theta_n}{\Gamma \Rightarrow \Theta},$$

where $\Gamma_1, \ldots, \Gamma_n, \Gamma$ *are g-context-pieces,* $\Theta_1, \ldots, \Theta_n, \Theta$ *are g-judgements (*$n = 0$ *is allowed).*

Then $ML_1^iW_T \vdash \Gamma \Rightarrow \Theta$ *is defined inductively by:*

If (Rule) is a rule of $ML_1^iW_T$ *as above,* Δ *is a g-context-piece such that* $\Delta, \Gamma_1, , \ldots,$ Δ, Γ_n, Δ, Γ *are* $g-contexts$, *and if* $ML_1^iW_T \vdash \Delta, \Gamma_i \Rightarrow \Theta_i$ *for* $i = 1, \ldots, n$, *then* $ML_1^iW_T \vdash \Delta, \Gamma \Rightarrow \Theta$.

Analogously we define $ML_1^eW_T \vdash \Gamma \Rightarrow \Theta$. $ML_1^eW_R \vdash \Gamma \Rightarrow \Theta$ *and* $ML_1^iW_R \vdash \Gamma \Rightarrow \Theta$ *is again defined analogously, but we refer to r-judgements, -contexts etc. instead of g-judgements, -contexts etc.*

The rules for $ML_1^iW_T$ *are listed as the general rules, the rules for intensional Martin-Löf's type theory and the intensional Tarski-rules for the universe.*

The rules for $ML_1^iW_R$ *are the general rules, the rules for intensional Martin-Löf's type theory and the intensional Russel-rules for the universe.*

The rules for $ML_1^eW_T$ *are the general rules, the rules for extensional Martin-Löf's type theory and the extensional Tarski-rules for the universe.*

The rules for $ML_1^eW_R$ *are the general rules, the rules for extensional Martin-Löf's type theory and the extensional Russel-rules for the universe.*

We will write Θ *for* $\Rightarrow \Theta$ *as a premise of a rule.*

Rules for Martin-Löf's type theory

In the following, let A, B *be g-types,* $a, b, r, s, t, r_i, s_i, t_i$ *be g-terms,* θ *be g-judgements,* Γ' *be a g-context-piece in the Tarski-versions,* A, B *be r-types,* $a, b, r, s, t, r_i, s_i, t_i$ *be r-terms,* θ *be r-judgements,* Γ' *be a r-context-piece in the Russel-versions. (So although we state that Tarski- and Russel-version have many rules in common, in fact the rules are different in the sense that for the Tarski-version we refer to g-objects, for the Russel-version we refer to r-objects). Further let* $x, y, z, u \in Var_{ML}$. *Additionally assume for all rules, that all substitution mentioned explicitly are allowed. For instance in the rule (*$N_{\overline{S}}^{=}$*), assume that* $s_1[x/t, y/P(t, s_0, (x, y)s_1)]$ *and* $A[z/St]$ *are allowed substitutions.*

General rules

(Common for all 4 versions of Martin-Löf's type theory)

$(ASS) \quad \frac{A \ type}{x{:}A \Rightarrow x{:}A}$

$(THIN) \quad \frac{A \ type \qquad \Gamma' \Rightarrow \Theta}{x{:}A, \Gamma' \Rightarrow \Theta}$

$(REFL) \quad \frac{t{:}A}{t{=}t{:}A} \qquad \frac{t{=}t{:}A}{t{:}A} \qquad \frac{A{=}A}{A \ type} \qquad \frac{A \ type}{A{=}A} \qquad \frac{t{:}A}{A \ type}$

$(SYM) \quad \frac{t{=}t'{:}A}{t'{=}t{:}A} \qquad \frac{A{=}B}{B{=}A}$

$(TRANS) \quad \frac{t{=}t'{:}A \qquad t'{=}t''{:}A}{t{=}t''{:}A} \qquad \frac{A{=}B \qquad B{=}C}{A{=}C}$

$(SUB) \quad \frac{x{:}A, \Gamma' \Rightarrow \Theta \qquad \Rightarrow t{:}A}{\Gamma'[x/t] \Rightarrow \Theta[x/t]}$

$(REPL1) \quad \frac{x{:}A, \Gamma' \Rightarrow B \ type \qquad \Rightarrow t{=}t'{:}A}{\Gamma'[x/t] \Rightarrow B[x/t]{=}B[x/t']}$

$(REPL2) \quad \frac{x{:}A, \Gamma' \Rightarrow s{:}B \qquad \Rightarrow t{=}t'{:}A}{\Gamma'[x/t] \Rightarrow s[x/t]{=}s[x/t']{:}B[x/t]}$

$(REPL3) \quad \frac{t{:}A \qquad A{=}B}{t{:}B} \qquad \frac{t{=}t'{:}A \qquad A{=}B}{t{=}t'{:}B}$

$(ALPHA) \quad \frac{x{:}A, \Gamma' \Rightarrow \theta}{x{:}A', \Gamma' \Rightarrow \theta} \qquad \frac{A \ type}{A{=}A'} \qquad \frac{t{:}A}{t{=}t'{:}A} \qquad (if\ A{=}_\alpha A',\ t{=}_\alpha t')$

Rules for intensional Martin-Löf's type theory

(rules are common for Russel- and Tarski-version)

Type introduction rules

$(N_k^T) \quad N_k \ type \quad (k \in I\!N)$

$(N^T) \quad N \ type$

$(\Pi^T) \quad \frac{x{:}A \Rightarrow B \ type}{\Pi x{\in}A.B \ type}$

$(\Sigma^T) \quad \frac{x{:}A \Rightarrow B \ type}{\Sigma x{\in}A.B \ type}$

$(W^T) \quad \frac{x{:}A \Rightarrow B \ type}{W x{\in}A.B \ type}$

$(+^T)$ $\frac{A\ type \quad B\ type}{A+B\ type}$

(I^T) $\frac{t:A \quad s:A \quad A\ type}{I(A,t,s)\ type}$

Introduction rules

(N_k^I) $n_k : N_k \quad (n < k,\ n,k \in I\!N)$

(N^I) $0 : N \quad \frac{t:N}{St:N}$

(Π^I) $\frac{x:A \Rightarrow t:B \quad x:A \Rightarrow B\ type}{\lambda x.t:\Pi x \in A.B}$

(Σ^I) $\frac{s:A \quad t:B[x/s] \quad x:A \Rightarrow B\ type}{p(s,t):\Sigma x \in A.B}$

(W^I) $\frac{s:A \quad t:B[x/s] \rightarrow Wx \in A.B \quad x:A \Rightarrow B\ type}{sup(s,t):Wx \in A.B}$

$(+^I)$ $\frac{s:A \quad A\ type \quad B\ type}{i(s):A+B} \quad \frac{s:B \quad A\ type \quad B\ type}{j(s):A+B}$

(I^I) $\frac{t=t':A}{\underline{r}:I(A,t,t')}$

Elimination rules

(N_k^E) $\frac{t:N_k \quad s_i:A[x/i_k](i=0...k-1) \quad x:N_k \Rightarrow A\ type}{C_k(t,s_0,...,s_{k-1}):A[x/t]} \quad (k \in I\!N)$

(N^E) $\frac{t:N \quad s_0:A[z/0] \quad x:N,y:A[z/x] \Rightarrow s_1:A[z/Sx] \quad x:N \Rightarrow A[z/x]\ type}{P(t,s_0,(x,y)s_1):A[z/t]}$

(Π^E) $\frac{t_0:\Pi x \in A.B \quad t_1:A \quad x:A \Rightarrow B\ type}{Ap(t_0,t_1):B[x/t_1]}$

(Σ^E) $\frac{r:\Sigma x \in A.B \quad x:A \Rightarrow B\ type}{p_0(r):A} \quad \frac{r:\Sigma x \in A.B \quad x:A \Rightarrow B\ type}{p_1(r):B[x/p_0(r)]}$

(W^E) $\frac{t_0:Wx \in A.B \quad x:A,y:(B \rightarrow Wx \in A.B),z:\Pi v \in B.C[u/Ap(y,v)] \Rightarrow t_2:C[u/sup(x,y)] \quad u:Wx \in A.B \Rightarrow C\ type}{R(t_0,(x,y,z)t_2):C[u/t_0]}$

$(+^E)$ $\frac{t_0:A+B \quad x:A \Rightarrow t_1:C[z/i(x)] \quad y:B \Rightarrow t_2:C[z/j(y)] \quad z:A+B \Rightarrow C\ type}{D(t_0,(x)t_1,(y)t_2):C[z/t_0]}$

(I^E) $\frac{t_0:I(A,t_1,t_2) \quad t_1:A \quad t_2:A}{t_1=t_2:A}$

Equality rules

$(N_k^=)$ $\frac{s_i:A[x/i_k](i=0...k-1) \quad x:N_k\Rightarrow A \ \mathit{type}}{C_k(n_k,s_0,...,s_{k-1})=s_n:A[x/n_k]}$ $(n < k,\ n,k \in I\!N)$

$(N_0^=)$ $\frac{s_0:A[z/0] \quad x:N,y:A[z/x]\Rightarrow s_1:A[z/Sx] \quad x:N\Rightarrow A[z/x] \ \mathit{type}}{P(0,s_0,(x,y)s_1)=s_0:A[z/0]}$

$(N_S^=)$ $\frac{t:N \quad s_0:A[z/0] \quad x:N,y:A[z/x]\Rightarrow s_1:A[z/Sx] \quad x:N\Rightarrow A[z/x] \ \mathit{type}}{P(St,s_0,(x,y)s_1)=s_1[x/t,y/P(t,s_0,(x,y)s_1)]:A[z/St]}$

$(\Pi^=)$ $\frac{\lambda x.t_0:\Pi x\in A.B \quad t_1:A \quad x:A\Rightarrow B \ \mathit{type}}{Ap(\lambda x.t_0,t_1)=t_0[x/t_1]:B[x/t_1]}$

$(\Sigma_0^=)$ $\frac{p(r,s):\Sigma x\in A.B \quad A \ \mathit{type}}{p_0(p(r,s))=r:A}$

$(\Sigma_1^=)$ $\frac{p(r,s):\Sigma x\in A.B \quad x\in A\Rightarrow B \ \mathit{type}}{p_1(p(r,s))=s:B[x/r]}$

$(\Sigma_2^=)$ $\frac{t:\Sigma x\in A.B}{t=p(p_0(t),p_1(t)):\Sigma x\in A.B}$

$(W^=)$ $\frac{r:A \quad s:(B[x/t_0]\rightarrow Wx\in A.B) \quad x:A,y:B\rightarrow Wx\in A.B,z:(\Pi v\in B.C[u/Ap(y,v)])\Rightarrow t:C[u/sup(x,y)] \quad u:Wx\in A.B\Rightarrow C \ \mathit{type}_{x:A\Rightarrow B} \ \mathit{type}}{R(sup(r,s),(x,y,z)t)=t[x/r,y/s,z/\lambda v.R(Ap(s,v),(x,y,z)t)]:C[u/sup(r,s)]}$

(If $v \notin FV(s) \cup FV((x,y,z)t)$*)*

$(+_0^=)$ $\frac{t_0:A \quad x:A\Rightarrow t_1:C[z/i(x)] \quad y:B\Rightarrow t_2:C[z/j(y)] \quad z\in A+B\Rightarrow C \ \mathit{type}}{D(i(t_0),(x)t_1,(y)t_2)=t_1[x/t_0]:C[z/i(t_0)]}$

$(+_1^=)$ $\frac{t_0:B \quad x:A\Rightarrow t_1:C[z/i(x)] \quad y:B\Rightarrow t_2:C[z/j(y)] \quad z\in A+B\Rightarrow C \ \mathit{type}}{D(j(t_0),(x)t_1,(y)t_2)=t_2[y/t_0]:C[z/j(t_0)]}$

$(I^=)$ $\frac{t_0:I(A,t_1,t_2) \quad A \ \mathit{type}}{t_0=\underline{r}:I(A,t_1,t_2)}$

Intensional Tarski rules for the universe

Tarski type introduction rules for the universe

(U^I) $U \ \mathit{type}$

(T^I) $\frac{a:U}{T(a) \ \mathit{type}}$ $\frac{a=a':U}{T(a)=T(a')}$

Tarski introduction rules for the universe

$(\underline{n}_k^I) \quad \underline{n}_k : U \quad (k \in I\!N)$

$(\underline{n}^I) \quad \underline{n} : U$

$(\pi^I) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{\pi x \in a.b:U}$

$(\sigma^I) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{\sigma x \in a.b:U}$

$(w^I) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{w x \in a.b:U}$

$(\widetilde{+}^I) \quad \frac{a:U \quad b:U}{a \widetilde{+} b:U}$

$(i^I) \quad \frac{a:U \quad t:T(a) \quad s:T(a)}{\widetilde{i}(a,t,s):U}$

Tarski equality rules for the universe

$(\underline{n}_k^=) \quad T(\underline{n}_k) = N_k \quad (k \in I\!N)$

$(\underline{n}^=) \quad T(\underline{n}) = N$

$(\pi^=) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{T(\pi x \in a.b) = \Pi x \in T(a).T(b)}$

$(\sigma^=) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{T(\sigma x \in a.b) = \Sigma x \in T(a).T(b)}$

$(w^=) \quad \frac{a:U \quad x:T(a) \Rightarrow b:U}{T(w x \in a.b) = W x \in T(a).T(b)}$

$(\widetilde{+}^=) \quad \frac{a:U \quad b:U}{T(a \widetilde{+} b) = T(a) + T(b)}$

$(i^=) \quad \frac{a:U \quad t:T(a) \quad s:T(a)}{T(\widetilde{i}(a,t,s)) = I(T(a),t,s)}$

Russel rules for the universe

Russel type introduction rules for the universe

$(U_R^I) \quad U \ \ type \quad \frac{A:U}{A \ \ type} \quad \frac{A=A':U}{A=A'}$

Russel introduction rules for the universe

$((N_k)^{U,I}_R) \quad N_k : U \quad (k < \omega)$

$(N^{U,I}_R) \quad N : U$

$(\Pi^{U,I}_R) \quad \frac{A:U \quad x:A \Rightarrow B:U}{\Pi x \in A.B:U}$

$(\Sigma^{U,I}_R) \quad \frac{A:U \quad x:A \Rightarrow B:U}{\Sigma x \in A.B:U}$

$(W^{U,I}_R) \quad \frac{A:U \quad x:A \Rightarrow B:U}{Wx \in A.B:U}$

$(+^{U,I}_R) \quad \frac{A:U \quad B:U}{A+B:U}$

$(i^{U,I}_R) \quad \frac{A:U \quad t:A \quad s:A}{i(A,t,s):U}$

Rules for extensional Martin-Löf's type theory

(Common for both $ML_1^e W_R$ and $ML_1^e W_T$)

Extensional type introduction rules

(N_k^T) *and* (N^T) *as before. Additional*

$(\Pi^{T,=})$ $\frac{A=A' \quad x:A\Rightarrow B=B'}{\Pi x\in A.B=\Pi x\in A'.B'}$

$(\Sigma^{T,=})$ $\frac{A=A' \quad x:A\Rightarrow B=B'}{\Sigma x\in A.B=\Sigma x\in A'.B'}$

$(W^{T,=})$ $\frac{A=A' \quad x:A\Rightarrow B=B'}{Wx\in A.B=Wx\in A'.B'}$

$(+^{T,=})$ $\frac{A=A' \quad B=B'}{A+B=A'+B'}$

$(I^{T,=})$ $\frac{A=A' \quad t=t':A \quad s=s':A}{I(A,t,s)=I(A',t',s')}$

Extensional Introduction rules

(N_K^I) *as before.*

$(N^{I,=})$ $0=0:N$ $\frac{t=t':N}{St=St':N}$

$(\Pi^{I,=})$ $\frac{x:A\Rightarrow t=t':B \quad x:A\Rightarrow B\ \mathit{type}}{\lambda x.t=\lambda x.t':\Pi x\in A.B}$

$(\Sigma^{I,=})$ $\frac{s=s':A \quad t=t':B[x/s] \quad x:A\Rightarrow B\ \mathit{type}}{p(s,t)=p(s',t'):\Sigma x\in A.B}$

$(W^{I,=})$ $\frac{s=s':A \quad t=t':B[x/s]\rightarrow Wx\in A.B \quad x:A\Rightarrow B\ \mathit{type}}{sup(s,t)=sup(s',t'):Wx\in A.B}$

$(+^{I,=})$ $\frac{s=s':A \quad A\ \mathit{type} \quad B\ \mathit{type}}{i(s)=i(s'):A+B}$ $\frac{s=s':B \quad A\ \mathit{type} \quad B\ \mathit{type}}{j(s)=j(s'):A+B}$

Extensional elimination rules

The rule intensional elimination rule (I^E) *and in addition:*

$(N_k^{E,=})$ $\frac{t=t':N_k \quad s_i=s_i':A[x/i_k](i=0...k-1) \quad x:N_k\Rightarrow A\ \mathit{type}}{C_k(t,s_0,...,s_{k-1})=C_k(t',s_0',...,s_{k-1}'):A[x/t]}$ $(k\in I\!N)$

$(N^{E,=})$ $\frac{t=t':N \quad s_0=s_0':A[z/0] \quad x:N,y:A[z/x]\Rightarrow s_1=s_1':A[z/Sx] \quad x:N\Rightarrow A[z/x]\ \mathit{type}}{P(t,s_0,(x,y)s_1)=P(t',s_0',(x,y)s_1'):A[z/t]}$

$(\Pi^{E,=})$ $\frac{t_0=t_0':\Pi x\in A.B \quad t_1=t_1':A \quad x:A\Rightarrow B\ \mathit{type}}{Ap(t_0,t_1)=Ap(t_0',t_1'):B[x/t_1]}$

$(\Sigma^{E,=})\quad \frac{r=r':\Sigma x\in A.B \quad x:A\Rightarrow B\ type}{p_0(r)=p_0(r'):A} \quad \frac{r=r':\Sigma x\in A.B \quad x:A\Rightarrow B\ type}{p_1(r)=p_1(r'):B[x/p_0(r)]}$

$(W^{E,=})\quad \frac{t_0=t_0':Wx\in A.B \quad x:A,y:B\rightarrow Wx\in A.B,z:\Pi v\in B.C[u/Ap(y,v)])\Rightarrow t_2=t_2':C[u/sup(x,y)] \quad u:Wx\in A.B\Rightarrow C\ type}{R(t_0,(x,y,z)t_2)=R(t_0',(x,y,z)t_2'):C[u/t_0]}$

$(+^{E,=})\quad \frac{t_0=t_0':A+B \quad x:A\Rightarrow t_1=t_1':C[z/i(x)] \quad y:B\Rightarrow t_2=t_2':C[z/j(y)] \quad z:A+B\Rightarrow C\ type}{D(t_0,(x)t_1,(y)t_2)=D(t_0',(x)t_1',(y)t_2'):C[z/t_0]}$

Extensional equality rules

All the intensional Equality rules, in addition

$(\Pi^{\eta})\quad \frac{t:\Pi x\in A.B}{\lambda x.Ap(t,x)=t:\Pi x\in A.B}\qquad$ *If* $x\notin FV(t)$

<u>Extensional Tarski rules for the universe</u>

Extensional Tarski type introduction rules for the universe

All the Tarski introduction rules for the universe.

Extensional Tarski introduction rules for the universe

The rules $(\underline{n}_k^I)$ *and* $(\underline{n}^I)$ *of intensional Tarski version* $ML_1^iW_T$*, in addition:*

$(\pi^{I,=})\quad \frac{a=a':U \quad x:T(a)\Rightarrow b=b':U}{\pi x\in a.b=\pi x\in a'.b':U}$

$(\sigma^{I,=})\quad \frac{a=a':U \quad x:T(a)\Rightarrow b=b':U}{\sigma x\in a.b=\sigma x\in a'.b':U}$

$(w^{I,=})\quad \frac{a=a':U \quad x:T(a)\Rightarrow b=b':U}{wx\in a.b=wx\in a'.b':U}$

$(\widetilde{+}^{I,=})\quad \frac{a=a':U \quad b=b':U}{a\widetilde{+}b=a'\widetilde{+}b':U}$

$(i^{I,=})\quad \frac{a=a':U \quad t=t':T(a) \quad s=s':T(a)}{\widetilde{i}(a,t,s)=\widetilde{i}(a',t',s'):U}$

Extensional Tarski equality rules for the universe

The equality rules of the intensional Tarski version $ML_1^iW_T$

<u>Extensional Russel rules for the universe</u>

Extensional Russel type introduction rules for the universe

All the Russel type introduction rules for the universe.

Extensional Russel introduction rules for the universe

$((N_k)_R^{U,I})$, $(N_R^{U,I})$ *as before*

$(\Pi_R^{U,I,=})\quad \frac{A=A':U \quad x:A\Rightarrow B=B':U}{\Pi x\in A.B=\Pi x\in A'.B':U}$

$(\Sigma_R^{U,I,=})\quad \frac{A=A':U \quad x:A\Rightarrow B=B':U}{\Sigma x\in A.B=\Sigma x\in A'.B':U}$

$(W_R^{U,I,=})\quad \frac{A=A':U \quad x:A\Rightarrow B=B':U}{Wx\in A.B=Wx\in A'.B':U}$

$(+_R^{U,I,=})\quad \frac{A=A':U \quad B=B':U}{A+B=A'+B':U}$

$(i_R^{U,I,=})\quad \frac{A=A':U \quad t=t':A \quad s=s':A}{I(A,t,s)=I(A',t',s'):U}$

Remark 2.6 *All the rules of* $ML_1^iW_T$ *are derived rules in* $ML_1^eW_T$*.*
All the rules of $ML_1^iW_R$ *are derived rules in* $ML_1^eW_R$*.*

Chapter 3

Comparison of the formulation à là Russel and the formulation à là Tarski

In this chapter we first prove, that the Russel-version contains the Tarski-version (lemma 3.3). We prove the converse (lemma 3.11) for a little bit extended versions of extensional Tarski- and Russel-Martin-Löf's type theory, which we define in 3.4. For the proof theoretical strength, this is sufficient, since we can embed the extended Tarski-version in KPi^+.

Definition 3.1 *(a) Define for C constructors,* $\phi(\underline{n}) := N$, $\phi(\underline{n}_k) := N_k$, $\phi(\pi) := \Pi$, $\phi(\sigma) := \Sigma$, $\phi(w) := W$, $\phi(\widetilde{i}) := I$, $\phi(\widetilde{+}) := +$, $\phi(C) := C$ *otherwise.*

(b) Define ϕ *: b-object* $\rightarrow$ *b-object by recursion on the b-object:*

$\phi(x) := x$ $(x \in Var_{ML})$,
$\phi(C(t_1, \ldots, t_n)) := \phi(C)(\phi(t_1), \ldots, \phi(t_n))$, $(C \neq T)$,
$\phi(T(t)) := \phi(t)$,
$\phi(\lambda x.t) := \lambda x.\phi(t)$.

(c) If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ *is a g-context-piece, then* $\phi(\Gamma) := x : \phi(A_1), \ldots, x_n : \phi(A_n)$.

(d) If r, s *are g-terms,* A *is a g-type, then* $\phi(A : \ type) := (\phi(A) : \ type)$, $\phi(r : A) := (\phi(r) : \phi(A))$, $\phi(r = s : A) := (\phi(r) = \phi(s) : \phi(A))$, $\phi(A = B) := (\phi(A) = \phi(B))$.

Lemma 3.2 *Assume* r, s, t, s_i *b-objects,* $x_i \in Var_{ML}$.

(a) $FV(t) = FV(\phi(t))$.

(b) If $t[x_1/s_1, \ldots, x_n/s_n]$ *is allowed, then* $\phi(t)[x_1/\phi(s_1), \ldots, x_n/\phi(s_n)]$ *allowed.*

(c) $\phi(t[x_1/s_1, \ldots, x_n/s_n]) = \phi(t)[x_1/\phi(s_1), \ldots, x_n/\phi(s_n)]$.

(d) If t is a g-term, -type, -judgement, -statement, -context, -context-piece, then $\phi(t)$ *is a r-term, -type, -judgement, -statement, -context, -context-piece.*

(e) $r =_\alpha s \rightarrow \phi(r) =_\alpha \phi(s)$

(f) If r *is a g-term and a r-term, then* $\phi(r) = r$.

If r *is a g-type and a r-type, then* $\phi(r) = r$.

Proof: (a) by induction on definition of b-objects, (b), (c) and (e) by (a) and the same induction, (d), (f) by induction on definition of g-objects.

Lemma 3.3 *If ML_T is the Tarski-version of extensional or intensional Martin-Löf's type theory, and ML_R is the corresponding Russel-version, then*

If $ML_T \vdash \Gamma \Rightarrow \theta$ then $ML_R \vdash \phi(\Gamma) \Rightarrow \phi(\theta)$.

Especially, if the statement is a statement both in ML_T and ML_R, then, if $ML_T \vdash \Gamma \Rightarrow \theta$ then $ML_R \vdash \Gamma \Rightarrow \theta$.

Proof:

Induction on the derivation. The general rules follow using lemma 3.2, all the rules, not dealing with the universe are immediate.

(U^I) is trivial, (T^I) is obvious. Introduction rules for the universe, e.g. (σ^I): By IH $ML_R \vdash \phi(\Gamma) \Rightarrow \phi(a) : U$, $ML_R \vdash \phi(\Gamma), x : \phi(b) \Rightarrow \phi(b) : U$, therefore $ML_R \vdash \phi(\Gamma) \Rightarrow \phi(\sigma x \in a.b) : U$, similarly follow the other rules or $(\sigma^{I,=})$. In the equality rules for the Universe, both sides of the conclusion are identical, and if the statement is $\Gamma \Rightarrow A = B$, we can derive $ML_R \vdash \phi(\Gamma) \Rightarrow \phi(A) :$ type, and by $(REFL)$ follows the assertion.

Definition 3.4 *(a) Let $ML_1^eW_{T,U}$ be defined as $ML_1^eW_T$, only with the additional rules*

(σ^E) $\quad \frac{\sigma x \in a.b:U}{a:U} \quad \frac{\sigma x \in a.b:U}{x:T(a) \Rightarrow b:U}$

(π^E) $\quad \frac{\pi x \in a.b:U}{a:U} \quad \frac{\pi x \in a.b:U}{x:T(a) \Rightarrow b:U}$

(w^E) $\quad \frac{w x \in a.b:U}{a:U} \quad \frac{w x \in a.b:U}{x:T(a) \Rightarrow b:U}$

$(\tilde{+}^E)$ $\quad \frac{a \tilde{+} b:U}{a:U} \quad \frac{a \tilde{+} b:U}{b:U}$

(i^E) $\quad \frac{\tilde{i}(a,s,t):U}{a:U} \quad \frac{\tilde{i}(a,s,t):U}{s:T(a)} \quad \frac{\tilde{i}(a,s,t):U}{t:T(a)}$

(b) Let $ML_1^eW_{R,U}$ be defined as $ML_1^eW_R$, only with the additional rules

(Σ_U^E) $\quad \frac{\Sigma x \in A.B:U}{A:U} \quad \frac{\Sigma x \in A.B:U}{x:A \Rightarrow B:U}$

(Π_U^E) $\quad \frac{\Pi x \in A.B:U}{A:U} \quad \frac{\Pi x \in A.B:U}{x:A \Rightarrow B:U}$

(W_U^E) $\quad \frac{W x \in A.B:U}{A:U} \quad \frac{W x \in A.B:U}{x:A \Rightarrow B:U}$

$(+_U^E)$ $\quad \frac{A+B:U}{A:U} \quad \frac{A+B:U}{B:U}$

(I_U^E) $\quad \frac{I(A,s,t):U}{A:U} \quad \frac{I(A,s,t):U}{s:A} \quad \frac{I(A,s,t):U}{t:A}$

Remark 3.5 *If $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow \theta$, then $ML_1^eW_{R,U} \vdash \phi(\Gamma) \Rightarrow \phi(\theta)$.*

Proof:

As 3.3.

Lemma 3.6 *In* $ML_1^iW_T$, $ML_1^eW_T$, $ML_1^eW_{T,U}$, $ML_1^iW_R$, $ML_1^eW_R$ *and* $ML_1^eW_{R,U}$

$$\frac{\Gamma, x : A, \Gamma' \Rightarrow \theta \quad \Gamma \Rightarrow A = A'}{\Gamma, x : A', \Gamma' \Rightarrow \theta}$$

is a rule, derived from the General rules only.

Proof:

Let y be a fresh variable. Then by $(THIN)$ $\Gamma, y : A', x : A, \Gamma' \Rightarrow \theta$, by (SYM), $(TRANS)$ $(REFL)$ $\Gamma \Rightarrow A'$ type, $\Gamma, y : A' \Rightarrow y : A'$ and $\Gamma, y : A' \Rightarrow y : A$, By (SUB) $\Gamma, y : A', \Gamma'[x/y] \Rightarrow \theta[x/y]$, by $(THIN)$ and (SUB) $\Gamma, x : A', \Gamma' \Rightarrow \theta$. (eventually we need some $(ALPHA)$, to make $\theta[x/y][y/x]$ is allowed and $\theta[x/y][y/x] = \theta$, the same for Γ'.

Definition 3.7 *(a) Define for* C *constructors,* $\psi(N) := \underline{n}$, $\psi(N_k) := \underline{n}_k$, $\psi(\Pi) := \pi$, $\psi(\Sigma) := \sigma$, $\psi(W) := w$, $\psi(I) := \tilde{i}$, $\psi(+) := \tilde{+}$, $\psi(C) := C$ *otherwise.*

(b) Define ψ *: b-object* $\rightarrow$ *b-object by recursion on the b-objects:*

$\psi(x) := x$ $(x \in Var_{ML})$,
$\psi(C(t_1, \ldots, t_n)) := \psi(C)(\phi(t_1), \ldots, \phi(t_n))$,
$\psi(\lambda x.t) := \lambda x.\psi(t)$.

(c) Define the function ρ *: b-object* $\rightarrow$ *b-object by recursion on the b-objects:*

$\rho(Sx \in r.s) := Sx \in \rho(r).\rho(s)$ $(S \in \Sigma, \Pi, W)$
$\rho(r + s) := \rho(r) + \rho(s)$,
$\rho(I(r, s, t)) := I(\rho(r), \psi(s), \psi(t))$,
$\rho(C) := C$ *for* $C \in \{N, N_k, U\}$,
$\rho(t) := T(\psi(t))$, *otherwise.*

(d) If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ *is a g-context-piece, then* $\rho(\Gamma) := x_1 : \rho(A_1), \ldots, x_n : \rho(A_n)$

(e) If r, s *are g-terms,* A *is a g-type, then* $\rho(A :\ \text{type}) := (\rho(A) :\ \text{type})$, $\phi(r : A) := (\psi(r) : \rho(A))$, $\rho(r = s : A) := (\psi(r) = \psi(s) : \rho(A))$, $\rho(A = B) := (\rho(A) = \rho(B))$.

(f) Define μ *: b-object* $\rightarrow$ *b-object by recursion on the b-objects:*

$\mu(T(sx \in r.s)) := \phi(s)x \in \mu(T(r)).\mu(T(s))$ $(s \in \{\sigma, \pi, w\})$,
$\mu(T(r\tilde{+}s)) := \mu(T(r)) + \mu(T(s))$,
$\mu(\tilde{i}(r, s, t))) := I(\mu(T(r)), s, t)$,
$\mu(T(C)) := \psi(C)$ *for* $C \in \underline{n}, \underline{n}_k$,
$\mu(Sx \in r.s) := Sx \in \mu(r).\mu(s)$ $(S \in \{\Sigma, \Pi, W\})$
$\mu(r + s) := \mu(r) + \mu(s)$,
$\mu(I(r, s, t)) := I(\mu(r), s, t)$,
$\mu(C) := C$ *for* $C \in \{N, N_k, U\}$,
$\mu(t) := t$, *otherwise.*

Lemma 3.8 *Assume* r, s, t, s_i *b-objects,* $x_i \in Var_{ML}$.

(a) $FV(t) = FV(\psi(t)) = FV(\rho(t)) = FV(\mu(t))$.

(b) If $t[x_1/s_1, \ldots, x_n/s_n]$ *allowed, then*
$\psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$,
$\mu(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$,
$\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$
are allowed.

(c) *If* t *is a r-term, then* $\psi(t)$ *is a g-term.*

If t *is a r-type, then* $\rho(t)$ *is a g-type.*

If t *is a g-type, then* $\mu(t)$ *is a g-type.*

If t *is a r-term, then* $\rho(t) = \mu(T(\psi(t)))$.

(d) *If* t, s_i *are b-objects, then* $\psi(t[x_1/s_1, \ldots, x_n/s_n]) = \psi(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)]$.

If t *is a g-type,* s_i *are g-terms, then* $\mu(\mu(t)) = \mu(t)$,

$\mu(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\mu(t)[x_1/s_1, \ldots, x_n/s_n])$,

If t *is a r-type,* s_i *are r-terms, then* $\rho(t[x_1/s_1, \ldots, x_n/s_n]) = \mu(\rho(t)[x_1/\psi(s_1), \ldots, x_n/\psi(s_n)])$.

(e) *If* t *is a* g *-judgement, -statement, -context, -context-piece, then* $\rho(t)$ *is a r-judgement, -statement, -context, -context-piece.*

(f) $r =_\alpha s \rightarrow \psi(r) =_\alpha \psi(s)$, $\rho(r) =_\alpha \rho(s)$, $\mu(r) =_\alpha \mu(s)$.

(g) *If* r *is a g-term and b-term, then* $\psi(r) = r$.

If r *is a g-type and b-type, then* $\rho(r) = r$.

Proof: (a) by induction on definition of b-objects, (b) by (a) and the same induction.
(c) follows by induction of b-objects, g-types, by the first two ones, and by induction on rterms.
(d): The first assertion is easy by induction on the definition of b-objects. The second is shown by induction on the definition of $\mu(t)$. The third follows again by induction, the fourth one follows using the third and first one.
(e), (f), (g) by induction on definition of the different objects.

Lemma 3.9 *Let* ML_T *be* $ML_1^i W_T$, $ML_1^e W_T$ *or* $ML_1^e W_{T,U}$ *and* Γ, Γ' *be g-context-pieces,* a, b, t *g-terms,* A, B *g-types,* x *a variable. The following applies:*

(a) *If* $ML_T \vdash \Gamma \Rightarrow s : A$, *or* $ML_T \vdash \Gamma \Rightarrow s = t : A$ *then* $ML_T \vdash \Gamma \Rightarrow A$ *type.*

(b) *If* $ML_T \vdash \Gamma \Rightarrow Sx \in A.B$ *type* ($S \in \{\Sigma, \Pi, W\}$), *then* $ML_T \vdash \Gamma \Rightarrow A$, $ML_T \vdash \Gamma, y : A \Rightarrow B[x/y]$ *type, for all* $y \in Var_{ML} \setminus X$ *for some finite set* X.

(c) *If* $ML_T \vdash \Gamma \Rightarrow A + B$ *type then* $ML_T \vdash \Gamma \Rightarrow A$ *type,* $ML_T \vdash \Gamma \Rightarrow B$ *type.*

(d) *If* $ML_T \vdash \Gamma \Rightarrow I(A, b, c)$ *type, then* $ML_T \vdash \Gamma \Rightarrow b : A$, $ML_T \vdash \Gamma \Rightarrow c : A$.

(e) *If* $ML_T \vdash \Gamma \Rightarrow T(b) :$ *type then* $ML_T \vdash \Gamma \Rightarrow b : U$.

(f) *If* $ML_T \vdash \Gamma \Rightarrow A = B$, *then* $ML_T \vdash \Gamma \Rightarrow A$ *type,* $ML_T \vdash \Gamma \Rightarrow B$ *type.*

(g) *If* $ML_T \vdash \Gamma, x : A, \Gamma' \Rightarrow \theta$ *then* $ML_T \vdash \Gamma \Rightarrow A$ *type.*

Proof: We first add the rules

$$(REPL1a) \quad \frac{x{:}A, \Gamma', \Gamma'' \Rightarrow B \text{ type} \quad \Rightarrow t = t' : A}{\Gamma'[x/t], \Gamma''[x/t'] \Rightarrow B[x/t'] \text{ type}}$$

and

$$(ALPHA1) \quad \frac{A \text{ type}}{A' \text{ type}} \quad \text{if } A =_\alpha A'$$

and in the extensional case and in $ML_1^eW_{T,U}$ additionally the intensional rules, and remove the $(REFL)$-rules

$$\frac{A = A}{A \text{ type}} \qquad \frac{t : A}{A \text{ type}}$$

If for this calculus the theorem is provable, then this calculus is equivalent to the original: If we have a proof in the original calculus, then embed it into the calculus, by applying, whenever we need the removed rules the weak inferences. If we have a proof in the new calculus, the result is a proof in the original calculus, since we only added derived rules.

Now induction on the length of the derivation:

In the cases (ASS), $(THIN)$, $(REFL)$, (SYM) and $(TRANS)$ follow trivially using the IH.

In (SUB) the difficulty are the the second conclusion in the cases (b): let the conclusion be for instance $\Gamma, \Gamma'[x/t] \Rightarrow (\Sigma x' \in A.B)[x/t]$. By IH $\Gamma, \Gamma', y : A' \Rightarrow B'[x'/y]$: type for $y \notin X$, therefore $\Gamma, \Gamma'[x/t], y : A'[x/t] \Rightarrow B'[x'/y][x/t]$ for $x \neq y$, $x \notin X$. (the substitution is allowed) Then for $y \notin X \cup \{x\}$, if $x = x'$ or $x \notin FV(B)$ follows $(\Sigma x' \in A.B)[x/t] = \Sigma x' \in A[x/t].B$ and we have the assertion, otherwise $x' \notin FV(t)$ and $B[x'/y][x/t] = B[x/t][x'/y]$.

In $(REPL1)$ the assertion follows since by IH we have $\Gamma \Rightarrow t : A$ and (SUB) or using $(REPL1a)$.

In $(REPL1a)$, we argue as in (SUB) .

In $(REPL2)$, the assumption follows by (SUB) and IH.

$(REPL3)$, $(ALPHA)$ follow by IH.

The type introduction rules and the introduction rules are immediate, in the elimination rules we need (SUB), and for (Σ_2^E) we need one application of (Σ_1^E), and in the equality rules we need the introduction rules and (SUB).

In the type introduction rules introduction rules and the equality rules for the universe nothing is to prove or the assertion is easy by IH.

In the extensional type introduction rules, for $(\Pi^{T,=})$, $(\Sigma^{T,=})$ and $(W^{T,=})$, the first part is easy by the intensional type introduction and IH for the second part we have by IH $x : A \Rightarrow B'$ type, and now use 3.6, (where we replace the argument using one of the removed rules by the IH), the extensional introduction rules are immediate, the extensional elimination rules follow as the introduction rules, (Π^η) follows by IH and in the extensional introduction rules there is nothing to prove. In the rules of 3.4 (a), there is nothing to prove.

Lemma 3.10 *(a)* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow r : U$ *, then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow T(r) = \mu(T(r))$

(b) $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow A$ *type , then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow A = \mu(A)$

Proof:

(a): Induction on the definition of r b-object. If for instance $ML_1^eW_T \vdash \Gamma \Rightarrow \sigma x \in a.b : U$, then by the rules of 3.4 $\Gamma \Rightarrow a : U$, $\Gamma, x : T(a) \Rightarrow b : U$, by IH $\Gamma \Rightarrow T(a) = \mu(T(a))$, $\Gamma, x : T(a) \Rightarrow T(b) = \mu(T(b))$, by extensional type introduction follows the assertion, similarly for the other terms, for which $\mu(T(t))$ does something.

(b) Induction on length of the type. Consider for instance the case $ML_1^eW_T \vdash \Gamma \Rightarrow \Pi x \in A.B$ type, then by 3.9 $\Gamma \Rightarrow A$ type, $\Gamma, y : A \Rightarrow B[x/y]$ type, by IH $\Gamma \Rightarrow A = \mu(A)$, $\Gamma, y : A \Rightarrow B[x/y] = \mu(B[x/y])$, by extensional type introduction and $(ALPHA)$ follows the assertion, similarly for the other types, but the $T(t)$-type.

If $\Gamma \Rightarrow T(t) : type$ by 3.9 we have $\Gamma \Rightarrow t : U$, and by the (a) follows the assertion.

Lemma 3.11 *If* $ML_1^eW_{R,U} \vdash \Gamma \Rightarrow \theta$ *then* $ML_1^eW_{T,U} \vdash \rho(\Gamma) \Rightarrow \rho(\theta)$.

Especially, if $\Gamma \Rightarrow \theta$ *is a statement of both* $ML_1^eW_{R,U}$ *and* $ML_1^eW_{T,U}$*, then we have: If* $ML_1^eW_{R,U} \vdash \Gamma \Rightarrow \theta$ *then* $ML_1^eW_{T,U} \vdash \Gamma \Rightarrow \theta$.

Proof: Induction on the derivation.

In most rules, the assertion follows by the same rules.

Difficult rules: (SUB), $(REPL)$: Use 3.8 (d), 3.10 and 3.6.

Equality rules and extensional equality rules: use for the substitution part the same argument.

Second and third rule in (U^I): we conclude $T(\psi(A))$, and using 3.10 and an easy argument follows the assertion.

Universe introduction rules (possibly extensional): easy.

Chapter 4

Definition of KPi

In this chapter we motivate Kripke-Platek set theory,and define the language and theory.

Kripke-Platek set theory is the result of omitting in axiomatic set theory the power set axiom and restricting the separation and collection axioms. In the ordinary separation axiom $\exists b.b = \{x \in a | \phi(x)\}$, if $\phi(x) = \exists y.\psi(x, y)$, the quantifier refers to all sets, even b itself — we have self-reference. If we restrict separation to Δ_0-formulas, e.g. $\phi = \exists y \in c.\psi(x, y)\}$ with ψ quantifier-free, the quantifiers refer only to subterms of ϕ (here c). Then we can assign an ordinal $level(b)$ to b in such a way, that the validity of $\phi(x)$ for $x \in a$ is determined by all sets of level less than b (in our case $level(b) = max\{level(a), level(c)\}$). We have no self-reference. The collection axiom need to be restricted to Δ_0-formulas as well. We can use the method called ramified set theory, to analyze the theory proof theoretically.

The resulting theory is $KP\omega$ which is examined in [Bar75](more precisely, Barwise examines KPU, KP with urelements, $KP\omega$ is KPU with no urelements) In order to have stronger theories, Jäger added in [Jäg79] a predicate $Ad(u)$ and axioms, expressing, that $Ad(u)$ means u is admissible, a transitive inner model of $KP\omega$, and further axioms claiming the existence of certain admissibles. The step to the next admissible corresponds to the step to the next inductive definition and will correspond in Martin-Löf's type theory to the building of one further nested W-type. In some of his theories, Jäger restricted the Δ_0 collection axiom and the foundation axiom, to have lower proof theoretical bounds.

We interpret $ML_1^e W_T$ in KPi^+, KPi_n^+, theories which are slight extensions of the Jäger's theory KPi, developed in [Jäg79] and analyzed proof theoretically in [Jäg83] and [JP82].

For further proof theoretical investigations of Kripke-Platek set theories, the reader might refer to [Jäg86]or [Poh82].

Definition 4.1 *Definition of Kripke-Platek set theory:*

(a) *Let L_{KP} be the classical first-order language, with terms being variables, atomic formulas being $u \in v$, $\neg(u \in v)$, $Ad(u)$, $\neg Ad(u)$. The set of Variables should be $Var_{KP} = \{u_i^{KP} | i \in N\}$ (a meta-set), $u_i^{KP} \neq u_j^{KP}$ for $i \neq j$.*
The formulas are built from atomic formulas by $\wedge$, $\vee$, $\forall$, $\exists$. We define $\neg A$ by the deMorgan's laws. The quantifier in $\forall x.\phi$ ($\exists x.\phi$) is bounded, if ϕ of the form $x \in v \rightarrow B$ ($x \in v \wedge B$) with $x \neq v$. A Δ_0-formula is a formula with no unbounded quantifier.
We abbreviate
$A \rightarrow B := ((\neg A) \vee B)$,
$\forall x \in v.B := \forall x.x \in v \rightarrow B$,
$\exists x \in v.B := \exists x.(x \in v \wedge B)$,

$(u = v) := ((\forall x \in u.x \in v) \wedge (\forall x \in v.x \in u))$,
$u \notin v := \neg(u \in v)$,
$tran(u) := \forall x \in u.\forall y \in x.y \in u$,
$infinite(u) := \exists x \in u.(x = x) \wedge \forall x \in u.\exists y \in u.x \in y$.
ψ a formula, then ψ^u means the replacing of every unbounded quantifier $\forall v$ by $\forall v \in u$ and $\exists v$ by $\exists v \in u$.

(b) *Definition of axiom schemes:*

(Ext)	$\forall x.\forall y.\forall z.x = y \rightarrow (x \in z \rightarrow y \in z) \wedge (Ad(x) \rightarrow Ad(y))$
$(Found)$	$\forall \vec{z}.[\forall x.(\forall y \in x.\phi(y, \vec{z}) \rightarrow \phi(x, \vec{z})) \rightarrow \forall x.\phi(x, \vec{z})]$
	(ϕ an arbitrary formula)
$(Pair)$	$\forall x.\forall y.\exists z.x \in z \wedge y \in z$.
$(Union)$	$\forall x.\exists z.\forall y \in x.\forall u \in y.u \in z$.
$(\Delta_0 - sep)$	$\forall \vec{z}.\forall w.\exists y.[\forall x \in y.(x \in w \wedge \phi(x, \vec{z})) \wedge \forall x \in w.\phi(x, \vec{z}) \rightarrow x \in y]$
	(ϕ a Δ_0-formula).
$(\Delta_0 - coll)$	$\forall \vec{z}.\forall w.[\forall x \in w.\exists y.\phi(x, y, \vec{z}) \rightarrow \exists w'.\forall x \in w.\exists y \in w'.\phi(x, y, \vec{z})]$
	(ϕ a Δ_0-formula).
$(Ad.1)$	$\forall x.Ad(x) \rightarrow tran(x) \wedge \exists w \in x.infinite(w)$.
$(Ad.2)$	$\forall x.\forall y.Ad(x) \wedge Ad(y) \rightarrow x \in y \vee x = y \vee y \in x$.
$(Ad.3)$	$\forall x.Ad(x) \rightarrow \psi^x$,
	(ψ an instance of (Pair), (Union), (Δ_0-sep), (Δ_0-coll)).
(Lim)	$\forall x.\exists y.Ad(y) \wedge x \in y$.
(inf)	$\exists x.infinite(x)$.
$(+)$	$\exists z.Ad(z) \wedge \forall x \in z.\exists y \in z.Ad(y) \wedge x \in y$.
$(+_n)$	$\exists x_1, \ldots, x_{n-1}, z.Ad(z) \wedge$
	$(\forall x \in z.\exists y \in z.Ad(y) \wedge x \in y) \wedge$.
	$Ad(x_1) \wedge \cdots \wedge Ad(x_{n-1}) \wedge$.
	$z \in x_1 \wedge x_1 \in x_2 \wedge \cdots \wedge\ x_{n-2} \in x_{n-1}$.

(c) *ES is the theory (Ext) + (Found) + (Pair) + (Union) + (Δ_0-sep) + (inf)*
KP is the theory (Ext) + (Found) + (Pair) + (Union) + (Δ_0-sep) + (Δ_0-coll)
$KP\omega$ is the theory ES + (Δ_0-coll) = KP + (inf)
KPl is the theory ES + (Ad.1 - 3) + (Lim) .
KPi is the theory ES + (Ad.1 - 3) + Lim + (Δ_0-coll)
(= KPl + $KP\omega$).
KPi^+ is the theory ES + (Ad.1 - 3) + (Lim) + (+).
KPi^+_n is the theory ES + (Ad.1 - 3) + (Δ_0-coll) + ($+_n$).

ES is a weak basic theory, KP is Kripke-Platek set theory without infinity axiom, $KP\omega$ is the usual Kripke-Platek set theory (without urelements but with infinity) KPl is a theory claiming the existence of infinitely many admissibles, but the model itself need not be an admissible. A model of KPl is is $L_{\Omega^{rec}_\omega}$. KPi formalizes the existence of infinitely many admissible, and the model itself need to be admissible. Its model is L_I, where I is an admissible fixed point of the enumeration of the admissible, the first recursive inaccessible. KPi^+ is a theory which formalizes the existence of I and and infinitely many admissibles above I and is modeled by $L_{\Omega^{rec}_{I+\omega}}$, and KPi^+_n expresses the existence of n admissibles above I, modeled by $L_{\Omega^{rec}_{I+n}}$.

Definition 4.2 *(a)* $pair(x, y) := \{\{x\}, \{x, y\}\}$,

$tripel(x, y) := pair(x, pair(x, y))$.

(We use this notation, to distinguish pair from the coding of pairs in the natural numbers).

(b) On should be the class or ordinals. ($a \in On$ is a Δ_1-predicate).

Part II

An upperbound for the proof theoretical strength of Martin-Löf's type theory

Chapter 5

Interpretation of terms and types

In this chapter, after an introduction, how to interpret Martin-Löf's type theory in Kripke-Platek set theory KPi^+, we formalize g-terms in it and introduce the reduction relation $\rightarrow_{red}$. (5.1), and prove some lemmata. Then we introduce the concept, how to interpret types as Σ functions. (5.7), and define the interpretation of the types, first for all but the Universe (5.8), and after that we give the definition of $\widehat{U}$, which is needed for the interpretation of U and $T(a)$ (5.9).

But let us start with an introduction in how to interpret Martin-Löf's type theory in KPi^+. The basic idea is to interpret a type as the set of those terms, for which we can prove in Martin-Löf's type theory, that they belong to the type: ($Term_{Cl}$ should be the set of closed g-terms.)

$$A \quad \mapsto A^* := \{t \in Term_{Cl} | ML_1^e W_T \vdash t : A\}$$

The problem is, that in this definition $t \in A^*$ does not give any information about the validity of the formula represented by A. We replace this definition by a set of terms, which are introduced by an introductory rule. An example might be ($A \times B = \Sigma x \in A.B$, where $x \notin B$, we take this example to avoid treating dependent types now)

$$A \times B \quad \mapsto (A \times B)^* := \{p(a,b) \in Term_{Cl} | a \in A \wedge b \in B\},$$

but since we have reduction of terms, we have to replace this definition by

$$(A \times B)^* := Compl(\{p(a,b) \in Term_{Cl} | a \in A \wedge b \in B\})$$

where

$$Compl(u) := \{r \in Term_{Cl} | \exists s \in u.s \text{ in normal-form } \wedge r \rightarrow_{red} s\}$$

The next task is to treat equality. We could treat this by saying, t and t' are equal if they have the same normal-form. But to make sure, that we do not get any nonsense like $t = t'$, where $t \rightarrow_{red} 0$, $t' \rightarrow_{red} S0$, we had to prove the existence of a unique normal-form, which is a technically very difficult area (note that we had to carry these proofs out in a restricted set theoretical framework), where the reduction rules have to be modified. We better try to avoid this problematic area, and use a far easier approach: We interpret types as sets of pairs of terms — the terms, which are considered to be equal.

A type is represented by a set of pairs of closed terms

An example would be

$$\begin{aligned}(A \times B)^* := \quad & Compl(\{pair(p(a,b), p(a',b')) \in Term_{Cl} \times Term_{Cl} | \\ & pair(a,a') \in A^* \wedge pair(b,b') \in B^*\}\end{aligned}$$

where now

$$Compl(u) := \{pair(r,r') \in Term_{Cl} \times Term_{Cl} | \exists s, s' \text{ in normal-form } . \\ r \rightarrow_{red} s \wedge s \rightarrow_{red} s' \wedge pair(s,s') \in u\}$$

Another example is

$$(A+B)^* := Compl(\{pair(i(r), i(r')) | pair(r,r') \in A^*\} \\ \cup \{pair(j(r), j(r')) | pair(r,r') \in B^*\})$$

or the easiest example

$$N^* := Compl(\{pair(S^n 0, S^n 0) | n \in \mathbb{N}\})$$

Now we want to interpret the type $A \rightarrow B$ ($= \Pi x \in A.B$ where x is a new variable). The problem is, that the introduction rules deduces from an open term a closed term: from $x : A \Rightarrow t : B$ we conclude $\lambda x.t : A \rightarrow B$. But the intended meaning of $x : A \Rightarrow t : B$ is, that, (as long as $x \notin FV(B)$) for every $s \in A^*$, $t[x/s] \in B^*$, and this is independent from the choice of equal elements of A^*: If $pair(s,s') \in A^*$, $pair(t[x/s], t[x/s']) \in B^*$. Putting these two definitions together in one, we can define

$$(A \rightarrow B)^* := Compl(\{pair(\lambda x.t, \lambda x'.t') | \forall pair(r,r) \in A^*.pair(t[x/r], t'[x'/r']) \in B^*\})$$

Now, the problem of confluence, we still have, is solved: The main reason for it is the use of open terms, which are part of the closed term $\lambda x.t$. But we have in our whole definition avoided the use of closed terms, and in the definition of $(A \rightarrow B)^*$, we see, that, if $pair(\lambda x.t, \lambda x.t) \in (A \rightarrow B)^*$, and if $t \rightarrow t'$ in a general sense for open terms, $t[x/r] \rightarrow_{red} t'[x/r]$ for closed terms r. But now, if B^* is closed under $\rightarrow_{red}$, we conclude $pair(\lambda x.t, \lambda x.t') \in (A \rightarrow B)^*$.

So we can define, that all terms $\lambda x.t$ are in normal form. To make the technicalities even more easier, we choose a deterministic reduction strategy, some kind of call-by-value strategy: we allow $Ap(\lambda x.r, s) \rightarrow_{red} r[x/s]$ only if s is in normal-form (note that $\lambda x.r$ is in normal-form), the same for all other constructors.

Now there is a new problem: Since we have dependent types, we have types containing free variables. Let us consider the Π-type:

$$(\Pi x \in A.B)^* := \\ Compl(\{pair(\lambda x.t, \lambda x'.t') | \forall pair(r,r') \in A^*.pair(t[x/r], t'[x'/r']) \in B^*[x/r]\})$$

(We will see later, that $t[x/r]$ is always allowed, as long as r is closed.)

So in fact, for a type with free variable x we could treat $B^*[x/r] := B[x/r]^*$. But we can not introduce the interpretation of all types, but only for finitely many types, since we are working in a restricted set theoretical system. The solution is, that we can interpret open types as Σ functions (see [Bar75]), and give some operation $A[x_1/r_1, \ldots, x_n/r_n]$ for Σ functions A and $x_i \in Var_{ML}$, r_i g-terms, in such a way, that the equality $B^*[x/r] = B[x/r]^*$ holds. Since we want to prove, that relation defined by the equality relation on the interpretation of a type is symmetric and transitive (we could prove this as part of the Main Lemma for all deduced types, but this would make this lemma even heavier), we add the additional condition $B^*[x/r] = B'^*[x/r']$ which will be fulfilled, whenever we have derived $x : A \Rightarrow B$ type:

$$(\Pi x \in A.B)^* := \\ Compl(\{pair(\lambda x.t, \lambda x'.t') | \forall pair(r,r') \in A^*.pair(t[x/r], t'[x'/r']) \in B^*[x/r] \\ \wedge B^*[x/r] = B^*[x/r']\})$$

Now original Martin-Löf's type theory has no fixed set of terms and types: they are introduced during the derivation of formulas. But we have extracted in chapter 2 a set of general terms, the g-terms, such that every term occurring in Martin-Löf's type theory is a g-term, but not vice versa, and a set of g-types. Now we can interpret every g-type as a Σ function. Up to this stage we do not know any correctness properties. This begins, in the last step, where we prove, that this interpretation is correct for derivations in Martin-Löf's type theory, in the sense that

$$ML_1^eW_T \vdash r \in A \quad \Rightarrow \quad KPi^+ \vdash pair(r,r) \in A^*,$$

or, considering a more complicated case,

$$(ML_1^eW_T \vdash x \in B \Rightarrow s = t \in A) \quad \Rightarrow$$

$$KPi^+ \vdash \forall pair(r,r') \in B^*.pair(s[x/r], t[x/r']) \in A^*[x/r].$$

The reader might have at this stage a first look at the following definitions, omitting all concepts dealing with the W-type and the universe, e.g. $W, U, T, (\pi x \in s.t), (\sigma x \in s.t)$ $(wx \in s.t)$, $\widetilde{i}(r,s,t), r\widetilde{+}s, \underline{n}, \underline{n}_k$, which will be explained next.

Let us now consider the W-type. From the rules follows, that an interpretation of it must be the least set closed under the applications of the W^I-rule. This set must be the least fixed point of the operator F, where, if we for simplicity switch back to the state, where types are interpreted as sets of terms rather than pairs of terms, F is defined as

$$F(u) := \{sup(r, \lambda x.s) | r \in A^* \wedge \forall t \in B^*[x/r].s[x/t] \in u\}$$

To get the least fixed point we have to iterate this operator up to the least admissible such that $A^* \in L_\alpha$ and $\forall pair(r,r') \in A^*.B^*[x/r] \in L_\alpha$ (in fact we will iterate it up to an ordinal such that even $\forall s \in Term_{Cl}.B^*[x/s] \in L_\alpha$).

The most powerful type of Martin-Löf's type theory is the Universe. When we consider a rule like

$$(\pi^I) \qquad \frac{a \in U \quad x \in T(a) \Rightarrow b \in U}{\pi x \in a.b : U}$$

we see, that we have to define simultaneously the elements of U and its interpretations as types $T(a)$. We will therefore first define a set $\widehat{U}$, which will be a set of tripels $tripel(a,A,b)$. The intended meaning of $tripel(a,A,b) \in \widehat{U}$ is, that a and b are equal elements in U and $T(a)^* = A$. This will be done in such a way, that, if we define

$$\sim := \{pair(a,b) \in Term_{Cl} \times Term_{Cl} | \exists A.tripel(a,A,b) \in \widehat{U}\}$$

and

$$f := \{pair(a,A) | \exists b \in Term_{Cl}.tripel(a,A,b) \in \widehat{U}\}$$

then $\sim$ is a symmetric and transitive relation, f is a function and $\forall a,b.a \sim b \rightarrow f(a) = f(b)$.

Having defined $\widehat{U}$, we can define $U^* := \sim$ and $T(a) := f(a)$

As for the W-type, $\widehat{U}$ will be the least fixed point of an operator $\widetilde{U}$. Since U is closed under the W-type, in the definition of $\widetilde{U}$ we have to go to the next admissible. So, to get $\widehat{U}$, we have to iterate $\widetilde{U}$ up to an admissible, which is closed under the formation of the next admissible, e.g. the first recursive inaccessible. This is the point, where we need the power of KPi^+. We need the existence of a recursive inaccessible, and in order to form W-types above U, we need the existence of infinitely many admissibles above this recursive inaccessible.

In fact, to interpret a particular proof of $ML_1^eW_T$, the nesting of W-types is limited, so we can interpret the types of this proof in KPi_n^+ for some n.

To prove in chapter 7, that all sentences provable in $ML_1^eW_T$ are provable in KPi_n^+ for some n we will add some constructors, having not necessary recursive reduction rules.

Definition 5.1 *(a) We add to the set of constructors a finite number of additional extended term constructors* $(A_i)_{i\in I}$ *with arities* $arity(A_i)$. *(*$I = \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}$*).*

The extended b-terms are defined as the b-terms, only using additionally the extended term constructors.

We extend substitution and α*-conversion for this extended system. Let* $(\lceil C\rceil)_{C \text{ constructor}}$ *Gödel-numbers for each constructor, such that we have a primitive recursive set* $TermConstr_{(A_i)_{i\in I}}$ *of these Gödel-numbers (we will usually omit the index* $(A_i)_{i\in I}$*), a function*

$$arity : TermConstr \to N,$$

and a primitive recursive function

$$n \mapsto \lceil C_n\rceil$$

(C_n *being the eliminator for the* N_n*-type).*

Further let $\lceil\lambda\rceil$, $\lceil Variable\rceil$, λ *be Gödelnumbers. (Gödelnumbers always means, that we have chosen different natural numbers).*

We interpret terms t *as follows:*

$\lceil z_i^{ML}\rceil :=< \lceil Variable\rceil, i >,$
$\lceil C(t_1, \ldots, t_n)\rceil :=< \lceil C\rceil, \lceil t_1\rceil, \ldots, \lceil t_n\rceil >,$
$\lceil \lambda z_i^{ML}.t\rceil :=< \lceil\lambda\rceil, i, \lceil t\rceil >.$

We have a primitive recursive set of all codes of extended b-terms $Term_{(A_i)_{i\in I}}$ *(again we omit usually the index), can break it primitive recursively into its part, can define a primitive recursive function* $sub : N \to N \to N$ *such that*

$$sub(\lceil t\rceil, << i_1, \lceil t_1\rceil >, \ldots, < i_n, \lceil t_n\rceil >>) = \lceil (t[z_{i_1}^{ML}/t_1, \ldots, z_{i_n}^{ML}/t_n])\rceil,$$

($< \cdot >$ *being the coding for sequences of natural numbers in natural numbers), a primitive recursive function* $FV' : N \to N$ *such that, if* $FV(t) = \{z_{i_1}^{ML}, \ldots, z_{i_n}^{ML}\}$ *with* $i_1 < \cdots < i_n$, $FV'(\lceil t\rceil) =< i_1, \ldots, i_n >$*, a primitive recursive definable subset* $Term_{Cl}$ *of indices of closed terms. In the following we will omit usually the Gödel-parenthesis* $\lceil\ \rceil$*, if it is clear that we are speaking of natural numbers.*

(b) The introductory term constructors are the term constructors 0, $\underline{r}$, $\underline{n}$, S, i, j, p, sup, $\widetilde{+}$, π, σ, w, $\widetilde{i}$.

(c) We assume, that for each A_i*, by which we extended our constructors, we have a function* $A_i^* : N^{arity(A_i)} \to N$*, defined in* KPi^+.

Let $\to_{red,imm_{(A_i,A_i^*)_{i\in I}}}$ *(again we will omit the index* $(A_i, A_i^*)_{i\in I}$*) be the set of pairs of closed terms, written* $t \to_{red,imm_{(A_i,A_i^*)}} t'$ *or short* $t \to_{red,imm} t'$ *for* $pair(t, t') \in \to_{red,imm}$*, defined as follows:*

If r, s, t, t', r_i are extended b-terms (and the left side of $\to_{red,imm}$ are closed terms), then
$p_0(p(r,s)) \to_{red,imm} r$,
$p_1(p(r,s)) \to_{red,imm} s$,
$Ap(\lambda x.r, s) \to_{red,imm} r[x/s]$,
$C_n(i_n, r_1, \ldots, r_n) \to_{red,imm} r_i$,
$D(i(r), s, t) \to_{red,imm} sr$,
$D(j(r), s, t) \to_{red,imm} tr$,
$P(0, s, t) \to_{red,imm} s$,
$P(Sr, s, t) \to_{red,imm} (trP(r,s,t))$, *(note that we write* rs *for* $Ap(r,s)$*),*
$R(sup(r,s), t) \to_{red,imm} (trs(\lambda z_i^{ML}.R(sz_i^{ML}, t)))$, *where* i *is minimal such that* $z_i^{ML} \notin FV(s) \cup FV(t)$
$A_i(S^{n_1}0, \ldots, S^{n_k}0) \to_{red,imm} S^{A_i^*(n_1,\ldots,n_k)}0$,
the pairs mentioned above are all pairs.
Note that it is primitive recursive decidable, if there exists a t' *such that* $t \to_{red,imm} t'$*, and for each term* t *there is at most one term* t' *such that* $t \to_{red,imm} t'$*, so* $\to_{red,imm}$ *represents are partial function.*

(d) *We define inductively a set of (indices for) terms in normal-form* $Term_{nf} \subset Term_{Cl}$*:*

If C *is an introductory* n*-ary term constructor,* $t_1, \ldots, t_n \in Term_{nf}$*, then* $C(t_1, \ldots, t_n) \in Term_{nf}$*.*
If C *is a* n*-ary term constructor (possibly an extended term constructor) that is not introductory,* $t_1, \ldots, t_n \in Term_{nf}$*, and there exists no* t *such that* $C(t_1, \ldots, t_n) \to_{red,imm} t$*, then* $C(t_1, \ldots, t_n) \in Term_{nf}$*.*
If $t \in Term$*,* $x \in Var_{ML}$*,* $FV(t) \subset \{x\}$*, then* $\lambda x.t \in Term_{nf}$*.*

We easily see, that $Term_{nf}$ *is a primitive recursively decidable relation.*

(e) *We define terms* $t \in Term_{Cl}$*, the next reduced term* t^{red}*.*
For $t \in Term_{nf}$*.* $t^{red} := t$*.*
If C *is a* n*-ary (possibly extended) term constructor,* $r_i \in Term_{Cl}$*,* $\exists i.r_i \notin Term_{nf}$*, then* $C(r_1, \ldots, r_n)^{red} := C(r_1^{red}, \ldots, r_n^{red})$*.*
If $t := C(r_1, \ldots, r_n) \notin Term_{nf}$*,* $r_i \in Term_{nf}$*, then* $t \to_{red,imm} t'$ *for some* t'*,* $t^{red} := t'$*.*

We define $r \to_{red} s :\leftrightarrow \exists n \in \omega, s \in \omega.sequence(s) \wedge lh(s) = n \wedge (s)_0 = r \wedge (s)_n = s \wedge \forall i < n.(s)_i^{red} = (s)_{i+1}$*.*

Lemma 5.2 *Assume* $x_i, y_i \in Var_{ML}$*.*

(a) $KPi^+ \vdash \forall r, s' \in Term.r =_\alpha r' \to FV(r) = FV(r')$*.*

(b) $KPi^+ \vdash \forall r, r_i \in Term.FV(r[\vec{x}/\vec{r}]) \subset FV(r) \cup \bigcup_{i=1}^n FV(r_i)$*.*

(c) *In* KPi^+ *we prove that for all* $s, t_i \in Term$*, all* $r_i \in Term_{Cl}$*,*
$s[\vec{x}/\vec{r}]$ *is allowed substitution* $\wedge$
$(s[\vec{y}/\vec{t}]$ *is allowed* $) \to s[\vec{x}/\vec{r}][\vec{y}/\vec{t}]$ *is allowed* $)\wedge$
$(s[\vec{x}/\vec{r}][\vec{y}/\vec{t}] = s[\vec{x}/\vec{r}, \vec{y}/\vec{t}])$*.*

(d) $KPi^+ \vdash \forall t, t' \in Term.\forall r_i, r_i' \in Term_{Cl}.(t =_\alpha t' \wedge r_i =_\alpha r_i') \to t[\vec{x}/\vec{r}] =_\alpha t'[\vec{x}/\vec{r}]$*.*

(e) $KPi^+ \vdash \forall x, x' \in Var_{ML}.\forall t, t' \in Term.\forall r, r' \in Term_{Cl}.$
$(\lambda x.t =_\alpha \lambda x'.t' \wedge r =_\alpha r') \to t[x/r] =_\alpha t'[x'/r']$

Proof:
(a), (b): easy.
(c): If $x \in Var_{ML}$ this is trivial, and if $s = C(t_1, \ldots, t_n)$ it follows by IH. If $s = \lambda x.t$, $[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{y}'/\vec{t}'] := [\vec{y}/\vec{t}] \setminus \{x\}$, $s[\vec{x}/\vec{r}] = \lambda x.(t[\vec{x}'/\vec{r}'])$ allowed, since $x \notin FV(r_i)$ and the IH. If $s[\vec{y}/\vec{t}]$ is allowed follows $t[\vec{y}'/\vec{t}']$ is allowed and $x \neq y \in FV(t) \to x \notin FV(t_i)$ therefore $t[\vec{x}'/\vec{r}'][\vec{y}'/\vec{t}']$ is allowed and $x \neq y \in FV(t[\vec{x}'/\vec{r}']) = FV(t) \to x \notin FV(t_i)$, $s[\vec{x}/\vec{r}][\vec{y}/\vec{t}]$ is allowed. Further follows $s[\vec{x}/\vec{r}][\vec{y}/\vec{t}] = \lambda x.(s[\vec{x}'/\vec{r}'][\vec{y}'/\vec{t}']) = \lambda x.(s[\vec{x}'/\vec{r}', \vec{y}'/\vec{t}']) = t[\vec{x}/\vec{r}, \vec{y}/\vec{t}]$.
(d) We prove first, that if $t =_\alpha t'$, $r_i \in Term_{Cl}$, $t[\vec{x}/\vec{r}] =_\alpha t'[\vec{x}/\vec{r}]$.
Induction on $t =_\alpha t'$. The case $t = t'$ is trivial, the cases of $t =_\alpha t'' =_\alpha t'$, $t = C(t_1, \ldots, t_n) \wedge t' = C(t'_1, \ldots, t'_n) \wedge t_i =_\alpha t'_i$ and $t =_N \lambda x.s \wedge t' =_N \lambda x.s' \wedge s =_\alpha s'$ follows by IH.
Case $t = \lambda x.s$, $t' = \lambda y.s[x/y]$, $y \notin FV(s)$, $t[x/y]$ is allowed, $[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x, y\}$. Then $y \notin FV(t)$, $t[\vec{x}/\vec{r}] = \lambda x.(s[\vec{x}'/\vec{r}'])$, by (c) therefore $s[\vec{x}'/\vec{r}'][x/y]$ is allowed, $y \notin FV(t[\vec{x}/\vec{t}]$, $s[\vec{x}'/\vec{r}'][x/y] = s[\vec{r}'/\vec{r}', x/y] = s[x/y][\vec{s}'/\vec{r}']$, $t'[\vec{x}/\vec{r}] = \lambda y.(s[\vec{x}'/\vec{r}'][x/y]) =_\alpha t[\vec{x}/\vec{r}]$.
Next we prove $r_i =_\alpha r'_i \to t'[\vec{x}/\vec{r}] =_\alpha t'[\vec{x}/\vec{r}']$, which follows by a trivial induction on the construction of t'.
(e) Induction on definition of $\lambda x.t =_\alpha \lambda x'.t'$: The case $\lambda x.t = \lambda x'.t'$ is trivial, if $\lambda x.t =_\alpha s =_\alpha \lambda x'.t'$, then $s = \lambda x''.t''$ and the assertion follows by IH, and if $x = x'$, $t =_\alpha t'$ the assertion follows by (d). If $t' = t[x/x']$, $x' \notin FV(\lambda x.t)$, $t[x/x']$ substitutable, $t'[x'/r'] =_\alpha t'[x'/r] = t[x/x'][x'/r] = t[x/r, x'/r] = t[x/r]$, since $x' \notin FV(t) \vee x' = x$.

Lemma 5.3 *(a)* $KPi^+ \vdash \forall r, s, s' \in Term_{Cl}.(r \to_{red} s \wedge r \to_{red} s') \to (s \to_{red} s' \vee s' \to_{red} s)$.

(b) $KPi^+ \vdash \forall r, s, s' \in Term_{Cl}.(r \to_{red} s \wedge r \to_{red} s' \wedge s, s' \in Term_{nf}) \to s = s'$.

(c) If C is a n-ary constructor, then

$$\begin{aligned} KPi^+ \vdash \quad & \forall r_1, \ldots, r_n, r'_1, \ldots, r'_n.(r_1 \to_{red} r'_1 \in Term_{nf} \wedge \cdots \wedge r_n \to_{red} r'_n \in Term_{nf}) \\ & \to C(r_1, \ldots, r_n) \to_{red} C(r'_1, \ldots, r'_n) \end{aligned}$$

(d) $KPi^+ \vdash \forall t, t', s \in Term_{Cl}.(t \to_{red} s \wedge t =_\alpha t') \to \exists s' \in Term_{Cl}.t' \to_{red} s'$.

(e) $KPi^+ \vdash \forall t, t' \in Term_{Cl}.t =_\alpha t' \to (t \in Term_{nf} \leftrightarrow t' \in Term_{nf})$.

Proof: (a), (b): Immediate, since we have chosen $t \mapsto t'^{red}$ as a function.
(c): Induction on maximum of the reduction sequences of $r_i \to_{red} r'_i$.
(d), (e): We show $t =_\alpha t' \to ((t \in Term_{nf} \leftrightarrow t' \in Term_{nf}) \wedge t^{red} \to_{red} t'^{red})$, by induction on the definition of $t =_\alpha t'$, side induction on construction on t:
If $t = t'$ the assertion is trivial, if $t =_\alpha t'' =_\alpha t'$ it follows by IH.
Case $t = C(t_1, \ldots, t_n)$. Then $t' = C(t'_1, \ldots, t'_n)$, $t_i =_\alpha t'_i$: If $t_i \notin Term_{nf}$, follows $t'_i \notin Term_{nf}$, $t_i^{red} =_\alpha t_i'^{red}$, $t^{red} = C(t_1^{red}, \ldots, t_n^{red}) =_\alpha t'$ by IH. If $t_i \in Term_{nf}$, $t \to_{red,imm} s$, $t' \to_{red,imm} s'$, where s' is just the same arrangement of the terms as in s, except that in the case of $C = R$ we might choose another variable z_i^{ML}, and in the case $C = Ap$ where we have to conclude using lemma 5.2 (d), and we have in all cases $s \to_{red} s'$. If $t_i \in Term_{nf}$, $\neg(t \to_{red,imm} s)$ we have $t, t' \in Term_{nf}$, $t^{red} = t =_\alpha t' = t'^{red}$.
If $t = \lambda x.s$, then $t' = \lambda y.s'$, $t, t' \in Term_{nf}$.

Definition 5.4 *(a)* $\alpha_I := min\{\gamma \in On | Ad(L_\gamma) \wedge \forall x \in L_\gamma.\exists y \in L_\gamma.Ad(y) \wedge x \in y\}$ *(definable in* KPi^+*),*

(b)

$$\begin{aligned} \alpha_{I,n} := \quad & min\{\gamma \in On | \exists \alpha_1, \ldots, \alpha_{n-1} \in On.Ad(L_{\alpha_1}) \wedge \cdots Ad(L_{\alpha_{n-1}}) \wedge \\ & \alpha_I \in \alpha_1 \wedge \alpha_1 \in \alpha_2 \wedge \cdots \wedge \alpha_{n-2} \in \alpha_{n-1} \wedge \gamma = \alpha_{n-1}\} \end{aligned}$$

(definable in KPi^+ or KPi^+_m for $n < m$)

(c) *If α is an ordinal, then $\alpha^+ := min\{\gamma \in On | Ad(L_\gamma) \wedge \alpha \in L_\gamma\}$ (definable in KPi^+),* $\alpha^{+,n} := min\{\gamma \in On | Ad(L_\gamma) \wedge \alpha \in L_\gamma\} \cup \{\alpha_{I,n}\}$ *(definable in KPi^+_n)*

(d) $a(u) := min\{c | Ad(c) \wedge u \in c\}$ *(definable in KPi^+).*
$\alpha(u) := \bigcup\{\gamma \in a(u) | \gamma \in On\}$, *definable in KPi^+.*
$a(u)^n := min\{c | Ad(c) \wedge u \in c\} \cup \{L_{\alpha_{I,n}}\}$, *definable in KPi^+_n.*
$\alpha(u)^n := \bigcup\{\gamma \in a(u)^n | \gamma \in On\}$, *definable in KPi^+_n.*

We need for the interpretation of W and U the transfinite iteration of an operator, defined as follows:

Definition 5.5 (a) *If F is an $(n+1)$-ary Σ function, let $F(\vec{n}, \cdot)$ be the Σ function, such that* $F(\vec{n}, \cdot)(u) := F(\vec{n}, u)$.

(b) *If F is a Σ function, we define by recursion on α*

$$F^\alpha(v) := \begin{cases} v & \text{if } \alpha = 0, \\ F(F^\beta(v)) & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta < \alpha, \beta \in On} F^\beta(v) & \text{if } \alpha \in Lim. \end{cases}$$

Definition 5.6 *We define the Σ function Compl*

$$Compl(a) := \{pair(r, s) \in Term_{Cl} \times Term_{Cl} | \exists r', s' \in Term_{nf}. \ r \longrightarrow_{red} r' \wedge s \longrightarrow_{red} s' \wedge pair(r', s') \in a\}$$

$Compl(a) \subset Term_{Cl} \times Term_{Cl}$, and will be used only, if $a \subset Term_{Cl} \times Term_{Cl}$.

Note that in fact this definition is relative to the extension of our set of constructors, so actually we have to write $Compl_{(A_i, A_i^*)_{i \in I}}$.

We will interpret each g-type occurring in a proof of Martin-Löf's type theory (note that there are only finitely many — we cannot give a general interpretation of all types) as Σ functions, with arguments represented by the free variables of the type. More precisely, if $FV(A) = \{z_1^{ML}, \ldots, z_n^{ML}\}$, ($z_i^{ML}$ as in the definition 2.1 of Var_{ML}) the arguments of the interpretation A^* will have arguments given by the variables $\{u_1^{KP}, \ldots, u_n^{KP}\}$ (u_i^{KP} as in definition 4.1 (a) of Var_{KP}). We introduce the following abbreviation:

Definition 5.7 *If A is a Σ function in KPi^+ with arguments represented by the free variables $\{u_{i_1}^{KP}, \ldots, u_{i_n}^{KP}\}$, $i_1 <, \ldots, i_n$, u_i^{KP} as in the definition 4.1 (a) of Var_{KP} z_i^{ML} as in the definition 2.1 of Var_{ML}, $r_1, \ldots, r_m$ extended b-objects, $j_k := min\{l | x_l = z_{j_k}^{ML}\}$ $(k = 1, \ldots, m)$, then*

$$A[x_1/r_1, \ldots, x_m/r_m] := A[u_{i_1}^{KP}/r_{j_1}, \ldots, u_{i_n}^{KP}/r_{j_n}],$$

where on the right-hand side we have the real substitution or (having introduced a Σ function symbol) the application of the function symbol to the arguments.

We will write $A[\vec{x}/\vec{n}]$ for $A[x_1/n_1, \ldots, x_n/n_n]$. Note that, if a variable occurs more than once in the sequence $x_1, \ldots, x_n$, only the first one is relevant.

Definition 5.8 *Definition of the interpretation of g-types A, namely A^*, which will be a Σ function, that has, if $FV(A) = \{z_1^{ML}, \ldots, z_n^{ML}\}$, ($z_i^{ML}$ as in definition 2.1 of Var_{ML}), arguments given by the variables $\{u_1^{KP}, \ldots, u_n^{KP}\}$ (u_i^{KP} as in defition 4.1 (a) of Var_{KP}). We will define it by giving the values $A^*[\vec{x}/\vec{s}]$.*

Additionally we define $lev(A) \in I\!N$.
The interpretation is relative to a choice of term constructors $(A_i)_{i\in I}$, *extending the given term constructors, together with some interpretation* A_i^*, *as stated in 5.1 (c), so in fact we have to write (we do not for simplicity)* $A^*_{(A_i,A_i^*)_{i\in I}}$.

We define in all, but the W-, U- *and* $T(t)$-*cases*

$$A^*[\vec{x}/\vec{s}] := Compl(A^{basis}[\vec{x}/\vec{s}]),$$

where A^{basis} *is defined as follows:*
For $k \in I\!N$,

$$N_k^{basis}[\,] := \{pair(n_k, n_k) | n < k\},$$

(note that we have a constant Σ *function)* $lev(N_k) := 0$

$$N^{basis}[\,] := \{pair(S^n 0, S^n 0) | n < \omega\},$$

$lev(N) := 0$
Let A, B *be g-types,* $m := max\{lev(A), lev(B)\}$.
Then $lev(\Pi x \in A.B) := lev(\Sigma x \in A.B) := m$, $lev(Wx \in A.B) := m+1$, *and we define:*

$$\begin{aligned}(\Pi x \in A.B)^{basis}[\vec{x}/\vec{s}] := \quad & \{pair(\lambda x.r, \lambda x'.r') \in Term_{nf} \times Term_{nf} | \\ & \forall pair(s,s') \in A^*[\vec{x}/\vec{s}].pair(r[x/s], r[x'/s']) \in B^*[x/s, \vec{x}/\vec{s}] \\ & \wedge B^*[x/s, \vec{x}/\vec{s}] = B^*[x/s', \vec{x}/\vec{s}]\},\end{aligned}$$

(more precisely we have to write:

$$\begin{aligned}(\Pi x \in A.B)^{basis}[\vec{x}/\vec{s}] := \quad & \{pair(t,t') \in Term_{nf} \times Term_{nf} | \\ & \quad \exists x, x' \in Var_{ML}, r, r' \in Term.t = \lambda x.r \wedge t' = \lambda x'.r' \wedge \\ & \quad \forall s, s' \in Term.pair(s,s') \in A^*[\vec{x}/\vec{s}] \rightarrow \\ & \qquad pair(r[x/s], r[x'/s']) \in B^*[x/s, \vec{x}/\vec{s}] \\ & \qquad \wedge B^*[x/s, \vec{x}/\vec{s}] = B^*[x/s', \vec{x}/\vec{s}]\},\end{aligned}$$

similarly in the following definitions)

$$\begin{aligned}(\Sigma x \in A.B)^{basis}[\vec{x}/\vec{s}] := \quad & \{pair(p(r,s), p(r',s')) \in Term_{nf} \times Term_{nf} | \\ & \quad pair(r,r') \in A^*[\vec{x}/\vec{s}] \wedge pair(s,s') \in B^*[x/r, \vec{x}/\vec{s}] \wedge \\ & \quad B^*[x/r, \vec{x}/\vec{s}] = B^*[x/r', \vec{x}/\vec{s}]\},\end{aligned}$$

$$Wx \in A.B^*[\vec{x}/\vec{s}] := F^{\alpha_{I,m}}(\vec{s}, \cdot)(\emptyset)$$

(note that $m = max\{lev(A), lev(B)\}$*)*
where
$F(\vec{s}, u) := Compl(F^{basis}(\vec{s}, u))$, *and*

$$\begin{aligned}& F^{basis}(\vec{s}, u) := \\ & \{pair(sup(r, \lambda x.s), sup(r', \lambda x'.s')) \in Term_{nf} \times Term_{nf} | pair(r,r') \in A^*[\vec{x}/\vec{s}] \wedge \\ & \qquad B[x/r, \vec{x}/\vec{s}] = B[x/r', \vec{s}] \\ & \qquad \wedge \forall pair(t,t') \in B^*[x/r, \vec{x}/\vec{s}].pair(s[x/t], s'[x'/t']) \in u\}\end{aligned}$$

If A, B are g-types, then $lev(A+B) := max\{lev(A), lev(B)\}$,

$$\begin{aligned}(A+B)^{basis}[\vec{x}/\vec{s}] := \quad & \{pair(i(r), i(r')) \in Term_{nf} \times Term_{nf} | pair(r, r') \in A^*[\vec{x}/\vec{s}]\} \\ & \cup \{pair(j(r), j(r')) \in Term_{nf} \times Term_{nf} | pair(r, r') \in B^*[\vec{x}/\vec{s}]\}\end{aligned}$$

If A is a g-type, s, t are g-terms, then $lev(I(A, s, t)) = lev(A)$ *and*

$$(I(A, r, s))^{basis}[\vec{x}/\vec{s}] := \{pair(\underline{r}, \underline{r}) | pair(r[\vec{x}/\vec{s}], s[\vec{x}/\vec{s}]) \in A^*[\vec{x}/\vec{s}]\}.$$

$lev(U) := 1$ *and*

$$U^*[\,] := \{pair(r, r') \in Term_{Cl} \times Term_{Cl} | \exists b \in TC(\widehat{U}) tripel(r, b, r') \in \widehat{U}\}.$$

($\widehat{U}$ will be defined in the next definition)
If t is a g-term, then $lev(T(t)) := 1$

$$T(t)^*[\vec{x}/\vec{s}] := \bigcup\{b \in TC(\widehat{U}) | tripel(t[\vec{x}/\vec{s}], b, t[\vec{x}/\vec{s}]) \in \widehat{U}\}.$$

Definition 5.9 *Definition of $\widehat{U}$:*
$\widehat{U} := \widetilde{U}^{\alpha_I}(\emptyset)$, *where $\widetilde{U}$ is a Σ function symbol, which can be defined in KPi^+ in the following way:*

Let
$Compl_U(a) := \{tripel(r, b, s) \in Term_{Cl} \times TC(a) \times Term_{Cl} | \exists r', s' \in Term_{nf}.$
$r \rightarrow_{red} r' \wedge s \rightarrow_{red} s' \wedge tripel(r', b, s') \in a\}.$

$$\widetilde{U}(u) := Compl_U(\widetilde{U}^{basis}(u)),$$

where

$$\begin{aligned}\widetilde{U}^{basis}(u) := \quad & \{tripel(\underline{n}_k, u_{fin}(k), \underline{n}_k) \in a(u) | k \in I\!N\} \\ \cup \quad & \{tripel(\underline{n}, u_{nat}, \underline{n})\} \\ \cup \quad & \{tripel(\pi x \in r.s, u_\pi(b, f), \pi x' \in r'.s') \in a(u) | \phi(r, x, s, r', x', s', b, f, u)\} \\ \cup \quad & \{tripel(\sigma x \in r.s, u_\sigma(b, f), \sigma x' \in r'.s') \in a(u) | \phi(r, x, s, r', x', s', b, f, u)\} \\ \cup \quad & \{tripel(wx \in r.s, u_w^{\alpha(u)}(b, f, \cdot)(\emptyset), wx' \in r'.s') \in a(u) | \\ & \quad \phi(r, x, s, r', x', s', b, f, u)\} \\ \cup \quad & \{tripel(r \widetilde{+} s, u_+(b, c), r' \widetilde{+} s') \in a(u) | \psi_+(r, s, r', s', b, c, u)\} \\ \cup \quad & \{tripel(\widetilde{i}(r, s, t), u_i(b, s, t), \widetilde{i}(r', s', t') \in a(u) | \psi_i(r, s, t, r', s', t', b, u)\}\end{aligned}$$

and

$$\begin{aligned}\phi(r, x, s, r', x', s', b, f, u) \quad := \quad & r, r' \in Term_{nf} \wedge s, s' \in Term \wedge f \in a(u) \\ & \wedge FV(s) \subset \{x\} \wedge FV(s') \subset \{x'\} \wedge tripel(r, b, r') \in u \wedge \\ & (\forall pair(t, t') \in b. tripel(s[x/t], f(t), s'[x'/t']) \in u)\end{aligned}$$

(note that $f(t) = \bigcup\{c \in TC(f) | pair(t, c) \in f\}$, so f need not be a function)

$$\psi_+(r, s, r', s', b, c, u) := r, s, r', s' \in Term_{nf} \wedge tripel(r, b, r') \in u \wedge tripel(s, c, s') \in u,$$

$$\begin{aligned}\psi_i(r, s, t, r', s', t', b, u) \quad := \quad & r, s, t, r', s', t' \in Term_{nf} \wedge tripel(r, b, r') \in u \wedge \\ & pair(s, s') \in b \wedge pair(t, t') \in b,\end{aligned}$$

$u_{fin}(k) : Compl(u_{fin}^{basis}(k), \ where$

$$u_{fin}^{basis}(k) := \{pair(n_k, n_k) | n < k\}$$

$u_{nat} := Compl(u_{nat}^{basis}), \ where$

$$u_{nat}^{basis} := \{pair(S^n 0, S^n 0) | n \in I\!N\}$$

$u_{\pi}(b, f) := Compl(u_{\pi}^{basis}(b, f)), \ where$

$$\begin{aligned} u_{\pi}^{basis}(b, f) := \quad & \{pair(\lambda x.t, \lambda x'.t') \in Term_{nf} \times Term_{nf} | \\ & \forall pair(r, r') \in b.pair(t[x/r], t'[x'/r']) \in f(r)\} \end{aligned}$$

$u_{\sigma}(b, f) := Compl(u_{\sigma}^{basis}(b, f)), \ where$

$$\begin{aligned} u_{\sigma}^{basis}(b, f) := \quad & \{pair(p(r, s), p(r', s')) \in Term_{nf} \times Term_{nf} | pair(r, r') \in b \wedge \\ & pair(s, s') \in f(r)\}, \end{aligned}$$

$u_w(b, f, v) := Compl(u_w^{basis}(b, f, v)), \ where$

$$\begin{aligned} u_w^{basis}(b, f, v) := \{ \quad & pair(sup(r, \lambda x.s), sup(r', \lambda x'.s')) \in Term_{nf} \times Term_{nf} | \\ & r, r' \in Term_{Cl}, s, s' \in Term, FV(s) \subset \{x\}, \ FV(s') \subset \{x'\}, \\ & pair(r, r') \in b, \forall pair(t, t') \in f(r).pair(s[x/t], s'[x'/t']) \in v)\} \end{aligned}$$

$u_{+}(b, c) := Compl(u_{+}^{basis}(b, c)), \ where$

$$\begin{aligned} u_{+}(b, c) \quad := \quad & \{pair(i(r), i(r')) \in Term_{nf} \times Term_{nf} | pair(r, r') \in b\} \\ \cup \quad & \{pair(j(r), j(r')) \in Term_{nf} \times Term_{nf} | pair(r, r') \in c\} \end{aligned}$$

$u_i(b, r, s) := Compl(u_i^{basis}(b, r, s)), \ where$

$$u_i^{basis}(b, r, s) := \{pair(\underline{r}, \underline{r}) | pair(r, s) \in b\}$$

Chapter 6

Correctness of the Interpretation

In this chapter, we show, that the interpretation is correct, in the sense, that, if $ML_1^e W_T \vdash r : A$, then $KPi^+ \vdash pair(r,r) \in A^*$. In chapter 7 we will then conclude, that this shows, that all arithmetical sentences provable in $ML_1^e W_T$, can be proven in KPi^+.

We first prove some correctness property: The relation defined by the type is a symmetric and transitive relation (lemma 6.10), we examine the relationship to substitution (lemma 6.12) and α-conversion (lemma 6.15). Further we introduce abbreviations, to state easily our Main Lemma (definitions 6.11 and 6.16). The heart of this chapter is the Main Lemma 6.18, which proves, that, if $ML_1^e W_T \vdash r : A$, then $KPi^+ \vdash pair(r, r,) \in A^*$. In fact we prove something more general, to have a stronger induction hypothesis. The proof is tedious, since we have to check the correctness of each rule, but not very complicated — the intuition for the upper bound is concentrated in. At the end we see, that we can even interpret an a little bit extended version $ML_1^e W_{T,U}$ in this way. This version is equivalent to $ML_1^e W_{R,U}$, an extension of $|ML_1^e W_R|$ and therefore we can interpret $ML_1^e W_R$ as well (lemma 6.19).

The next lemma shows, that by iterating $u_w(b, f, \cdot)$ up to the next admissible, we get a fixed point.

Lemma 6.1 *(a)* $\forall \gamma < \delta. u_w^\gamma(b, f, \cdot)(\emptyset) \subset u_w^\delta(b, f, \cdot)(\emptyset)$.

(b) $(b \in a \wedge f \in a \wedge Ad(a) \wedge \alpha = \bigcup_{\delta \in a \cap On} \delta) \rightarrow$
$\forall \gamma > \alpha. u_w^\gamma(b, f, \cdot)(\emptyset) = u_w^\alpha(b, f, \cdot)(\emptyset)$.

(c) The proof can be done in KPi^+.

Proof: (a) follows by induction on δ.

(b) It is sufficient to show, with $u' := u_w^\alpha(b, f, \cdot)(\emptyset)$, that $u_w(b, f, u') \subset u'$. (Then the assertion follows immediately by induction on γ)

Since $Compl(u') \subset u'$ it is sufficient to prove $u_w^{basis}(b, f, u') \subset u'$. Now, if

$$pair(sup(r, \lambda x.s), sup(r, \lambda x'.s')) \in u_w^{basis}(b, f, u')$$

follows

$$\forall t, t' \in Term_{Cl}. pair(t, t') \in f(r) \rightarrow \exists \delta < \alpha. pair((s[x/t], s'[x'/t']) \in u_w^\delta(b, f, u))$$

Since $Ad(a)$ (here we need (Δ_0-coll), there exists $\rho < \alpha$, such that

$$\forall t, t' \in Term_{Cl}. pair(t, t') \in f(r) \rightarrow \exists \delta < \rho. pair((s[x/t], s'[x'/t']) \in u_w^\delta(b, f, u))$$

Now follows

$$pair(sup(r, \lambda x.s),\ sup(r', \lambda x'.s')) \in u_w^{\rho+1}(b, f, u) \subset u'$$

and the assertion.

Definition 6.2 *(a)* $equiv(u) :\Leftrightarrow$
$\forall r, s, t, r', s' \in Term_{Cl}.(pair(r, s) \in u \to pair(s, r) \in u) \wedge$
$((pair(r, s) \in u \wedge pair(s, t) \in u) \to pair(r, t) \in u)$.
(note that we do not claim reflexivity)

(b) $Cor(u) :\Leftrightarrow$
$\forall r, r', r'' \in Term_{Cl}.\forall b, b'.(tripel(r, b, r') \in u \to [tripel(r', b, r) \in u \wedge equiv(b)$
$\wedge [tripel(r', b', r'') \in u \to (tripel(r, b, r'') \in u \wedge b = b')]])$

Remark 6.3 *(a)* $(Cor(u) \wedge tripel(r, b, r') \in u \wedge tripel(r, b', r'') \in u) \to (b = b' \wedge tripel(r, b, r) \in u)$.

(b) If $Cor(u)$ *and*

$$\sim := \{pair(a, b) \in Term_{Cl} \times Term_{Cl} | \exists A \in TC(u).tripel(a, A, b) \in u\}$$

and

$$f := \{pair(a, A) \in Term_{Cl} \times TC(u) | \exists b \in Term_{Cl}.tripel(a, A, b) \in u\}$$

then $\sim$ *is a symmetric and transitive relation,* f *is a function and* $\forall a, b.a \sim b \to f(a) = f(b)$.

Proof: Obvious.

Lemma 6.4 *Assume* $r, s, t, r', s', t' \in Term$, $x, x' \in Var_{ML}$, b, f, u *sets.*

(a) $(\phi(r, x, s, r', x', s', b, f, u) \wedge Cor(u)) \to$
$(b \in a(u) \wedge \exists f \in a(u).\forall pair(t, t') \in b.f(t) = f(t') = f'(t) = f'(t'))$.

(b) $\psi_+(r, s, r', s', b, c, u) \to b, c \in a(u)$

(c) $\psi_i(r, s, t, r', s', t', b, u) \to b \in a(u)$

Proof:
(a) $b \in TC(u) \in a(u)$.
Let $f' := \{pair(t, c) \in Term_{Cl} \times TC(u) | pair(t, t) \in b \wedge tripel(s[x/t], c, s[x/t]) \in u\}$.
$f' \in a(u)$. Further, if $pair(t, t') \in b$ follows $tripel(s[x/t], f(t), s'[x/t']) \in u$, by $Cor(u)$ $tripel(s[x/t], f(t), s[x/t]) \in u$, further $\forall c.tripel(s[x/t], c, s'[x/t]) \in u \to c = f(t)$, f' is a function, $f(t) = f'(t)$ and by

$$tripel(s'[x/t], f(t), s[x/t]) \in u \wedge tripel(s'[x/t], f(t'), s[x/t']) \in u$$

follows $f(t) = f(t')$.
(b), (c) follow by $b, c \in TC(u)$.

Lemma 6.5 *Assume* $r, s, t, r', s', t' \in Term$, $x, x' \in Var_{ML}$, b, f, u *sets.*

(a) $\forall k \in I\!N.tripel(\underline{n}_k, u_{fin}(k), \underline{n}_k) \in a(u)$.

(b) $tripel(\underline{n}, u_{nat}, \underline{n}) \in a(u)$.

(c) $(Cor(u) \wedge \phi(r, x, s, r', x', s', b, f, u)) \to$
$(tripel(\pi x \in r.s, u_\pi(b, f), \pi x' \in r'.s'),\ tripel(\sigma x \in r.s, u_\sigma(b, f), \sigma x' \in r'.s') \in a(u) \wedge$
$tripel(wx \in r.s, u_w^{\alpha(u)}(b, f, \cdot)(\emptyset), wx' \in r'.s') \in a(a(u)))$.

(d) $(Cor(u) \wedge \psi_+(r,s,r',s',b,c,u)) \to tripel(r\widetilde{+}s, u_+(b,c), r\widetilde{+}s) \in a(u)$.

(e) $(Cor(u) \wedge \psi_i(r,s,t,r',s',t',b,u)) \to tripel(\widetilde{i}(r,s,t), u_i(b,s,t), \widetilde{i}(r',s',t')) \in a(u)$.

Proof: By 6.4

Lemma 6.6 *Assume* $r,s,t,r',s',t',r'',s'',t'' \in Term$, $x,x',x'' \in Var_{ML}$, b,b',f,f',u,u' *sets.*

(a) $\phi(r,x,s,r',x',s',b,f,u) \to Cor(u) \to \phi(r',x',s',r,x,s,b,f,u)$.

(b) $(\phi(r,x,s,r',x',s',b,f,u) \wedge \phi(r',x',s',r'',x'',s'',b',f',u') \wedge Cor(u \cup u')) \to$
$\phi(r,x,s,r'',x'',s'',b,f,u \cup u') \wedge b = b' \wedge$
$\forall pair(t,t') \in b.f(t) = f'(t) = f(t') = f'(t')$

(c) $\psi_+(r,s,r',s',b,c,u) \to Cor(u) \to \psi_+(r',s',r,s,b,c,u)$.

(d) $(\psi_+(r,s,r',s',b,c,u) \wedge \psi_+(r',s',r'',s'',b',c',u') \wedge Cor(u \cup u') \to$
$(\psi_+(r,s,r'',s'',b,c,u \cup u') \wedge b = b' \wedge c = c'))$

(e) $\psi_i(r,s,t,r',s',t',b,u) \to Cor(u) \to \psi_i(r',s',t',r,s,t,b,u)$.

(f) $(\psi_i(r,s,t,r',s',t',b,u) \wedge \psi_i(r',s',t',r'',s'',t'',b',u') \wedge Cor(u \cup u') \to$
$(\psi_i(r,s,t,r'',s'',t'',b,u \cup u') \wedge b = b'))$

Proof:

(a) $tripel(r',b,r) \in u$. If $pair(t,t') \in b$, then

$$tripel(s[x/t'], f(t'), s'[x'/t]) \in u,\ tripel(s'[x'/t], f(t'), s[x/t']) \in u,$$

$f(t) = f(t')$ by 6.4 (a).

(b) $u'' := u \cup u'$. $b = b'$ by $Cor(u'')$. $tripel(r,b,r'') \in u''$. If $pair(t,t') \in b$, follows

$$tripel(s[x/t], f(t), s'[x'/t]) \in u'',\ tripel(s'[x'/t], f(t), s''[x''/t']) \in u'',$$

therefore $tripel(s[x/t], f(t), s''[x''/t']) \in u''$. Further

$$tripel(s'[x'/t], f(t), s[x/t]) \in u'',\ tripel(s'[x'/t], f'(t), s''[x''/t]) \in u'',$$

therefore $f(t) = f(t')$

(c) - (f): Easy.

Lemma 6.7 *Assume* $r,s,t,r',s',t' \in Term$, $x,x' \in Var_{ML}$, b,f,u *sets.*

(a) $\forall k \in I\!N.equiv(u^{basis}_{fin}(k))$.

(b) $equiv(u^{basis}_{nat})$.

(c) $(Cor(u) \wedge \phi(r,x,s,r',x',s',b,f,u)) \to (equiv(u^{basis}_{\pi}(b,f))equiv(u^{basis}_{\sigma}(b,f)) \wedge$
$\forall v.equiv(v) \to equiv(u^{basis}_{w}(b,f,v)))$.

(d) $(Cor(u) \wedge \psi_+(r,s,r',s',b,c,u)) \to equiv(u^{basis}_{+}(b,c))$

(e) $(Cor(u) \wedge \psi_i(r,s,t,r',s',t',b,u)) \to equiv(u^{basis}_{i}(b,s,t))$

Proof:

(a), (b) are trivial.

(c): Assume $pair(\lambda x.t, \lambda x'.t') \in u_\pi^{basis}(b, f)$, $pair(\tilde{r}, \tilde{r}') \in b$. Then $pair(\tilde{r}', \tilde{r}) \in b$, $pair(t[x/\tilde{r}'], t'[x/\tilde{r}]) \in f(\tilde{r}')$, $pair(t'[x/\tilde{r}], t[x/\tilde{r}']) \in f(\tilde{r}') = f(\tilde{r})$ so we have symmetry of $u_\pi^{basis}(b, f)$, similarly for transitivity and u_σ^{basis}, u_w^{basis}.

(d)- (e) are easy.

Lemma 6.8 *(a)* $(Cor(u) \wedge (\forall tripel(a, b, a') \in u.a, a' \in Term_{nf})) \to Cor(Compl_U(u))$.

(b) $(equiv(u) \wedge u \subset Term_{nf} \times Term_{nf}) \to equiv(Compl(u))$.

Proof:

By lemma 5.3 (b).

Lemma 6.9 *(a)* $Cor(u) \to Cor(\widetilde{U}(u))$,

(b) $u \subset u' \wedge Cor(u') \to \widetilde{U}(u) \subset \widetilde{U}(u')$,

(c) $Cor(\widehat{U})$.

Proof: (a): by 6.6, 6.7, 6.8.

(b) The only difficulty is, to show that

$$(\phi(r, x, s, r', x', s', b, f, u) \wedge \phi(r, x, s, r', x', s', b, f, u')) \to u_w^{\alpha(u)}(b, f, \cdot)(\emptyset) = u_w^{\alpha(u')}(b, f, \cdot)(\emptyset).$$

This follows by 6.1.

(c) Follows by (a), (b).

We conclude, that the interpretation of each g-types gives a symmetric and transitive relation:

Lemma 6.10 *If A g-type, then* $KPi^+ \vdash \forall s_1, \ldots, s_n \in Term_{Cl}.equiv(A^*[\vec{x}/\vec{s}])$

Proof: Induction on the definition of types.

In the cases $(\Pi x \in A.B)^*$ we need the additional condition in the definition of the interpretation $B^*[x/s, \vec{x}/\vec{s}] = B^*[x/s', \vec{x}/\vec{s}]$, similarly for Σ, W.

The difficult cases are $equiv(U^*)$, which follows from $Cor_U(u)$, and $equiv(T(t))^*[\vec{r}]$, which is trivial if $\neg\exists b.tripel(t^*[\vec{r}], b, t^*[\vec{r}]) \in \widehat{U}$, and if $tripel(t^*[\vec{r}], b, t^*[\vec{r}]) \in \widehat{U}$ for some b, follows $T(t)^*[\vec{r}] = b$, $equiv(b)$.

Now we define the interpretation of the judgements in the L_{KP}.

Definition 6.11 *Let* A, B *g-types,* s, t *g-terms,* $FV(A)$, $FV(B)$, $FV(s)$, $FV(t) \subset \{x_1, \ldots, x_n\}$, $r_1, \ldots, r_n$, $s_1, \ldots, s_n$ *be extended g-terms.*

(a) $(A\ type)^*[\vec{x}/\vec{r}; \vec{s}] :\Leftrightarrow (A\ type)^*[x_1/r_1; s_1, \ldots, x_n/r_n; s_n] :\Leftrightarrow (A^*[\vec{x}/\vec{r}] = A^*[\vec{x}/\vec{s}])$.

(b) $(A = B)^*[\vec{x}/\vec{r}; \vec{s}] :\Leftrightarrow (A = B)^*[x_1/r_1; s_1, \ldots, x_n/r_n; s_n] :\Leftrightarrow (A^*[\vec{x}/\vec{r}] = B^*[\vec{x}/\vec{s}])$.

(c) $(t : A)^*[\vec{x}/\vec{r}; \vec{s}] :\Leftrightarrow (t : A)^*[x_1/r_1; s_1, \ldots, x_n/r_n; s_n] :\Leftrightarrow pair(t[\vec{x}/\vec{r}], t[\vec{x}/\vec{s}]) \in A^*[\vec{x}/\vec{r}]$.

(d) $(t = t' : A)^*[\vec{x}/\vec{r}; \vec{s}] :\Leftrightarrow (t = t' : A)^*[x_1/r_1; s_1, \ldots, x_n/r_n; s_n] :\Leftrightarrow$ $pair(t[\vec{x}/\vec{r}], t'[\vec{x}/\vec{s}]) \in A^*[\vec{x}/\vec{r}]$.

We will not mention the variables $x_1, \ldots, x_n$ *explicitly, if they are the variables, mentioned in the context, writing* $(A = B)^*[\vec{r}, \vec{s}]$, $(t : A)^*[\vec{r}; \vec{s}]$, $(t = t' : A)^*[\vec{r}; \vec{s}]$.

Lemma 6.12 *(Substitution lemma).*

Let C, D *be g-types,* r, s, t_i, t_i' *g-terms,* $x_i, y_i \in Var_{ML}$. *Then:*

(a) *If* $r[\vec{x}/\vec{t}]$ *is an allowed substitution,* $FV(r[\vec{x}/\vec{t}]) \subset \{y_1, \dots, y_n\}$*, then*
$KPi^+ \vdash \forall \vec{r} \in Term_{Cl}.r[\vec{x}/\vec{t}][\vec{y}/\vec{r}] = r[x_1/t_1[\vec{y}/\vec{r}], \dots, x_n/t_n[\vec{y}/\vec{r}], \vec{y}/\vec{r}]$.
(Note that, if variables occur more than once in $[\vec{y}/\vec{r}]$*, only the first substitution is relevant.)*

(b) *If* $C[\vec{x}/\vec{t}]$ *is an allowed substitution,* $FV(C[\vec{x}/\vec{t}]) \subset \{y_1, \dots, y_n\}$*, then*
$KPi^+ \vdash \forall \vec{r}, r' \in Term_{Cl}.C[\vec{x}/\vec{t}]^*[\vec{y}/\vec{r}] = C^*[x_1/t_1[\vec{y}/\vec{r}], \dots, x_n/t_n[\vec{y}/\vec{r}], \vec{y}/\vec{r}]$.

(c) *If* $A[\vec{x}/\vec{t}]$, $B[\vec{x}/\vec{t'}]$ *are allowed substitutions,* $FV(A[\vec{x}/\vec{t}]), FV(B[\vec{x}/\vec{t'}]) \subset \{y_1, \dots, y_n\}$*, then*
$KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(A = B)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]); (\vec{t'}[\vec{x}/\vec{s}]), \vec{x}/\vec{r}; \vec{s}] \leftrightarrow (A[\vec{x}/\vec{t}] = B[\vec{x}/\vec{t'}])^*[\vec{x}/\vec{r}; \vec{s}]$.

(d) *If* $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}]$ *are allowed substitutions,* $FV(A[\vec{x}/\vec{t}]), FV(r[\vec{x}/\vec{t}]) \subset \{y_1, \dots, y_n\}$*, then*
$KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(r : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]); (\vec{t}[\vec{x}/\vec{s}]), \vec{x}/\vec{r}; \vec{s}] \leftrightarrow (r[x/t] : A[x/t])^*[\vec{x}/\vec{r}; \vec{s}]$.

(e) *If* $A[\vec{x}/\vec{t}]$, $r[\vec{x}/\vec{t}], s[\vec{x}/\vec{t'}]$ *are allowed substitutions,* $FV(A[\vec{x}/\vec{t}])$, $FV(r[\vec{x}/\vec{t}])$, $FV(s[\vec{x}/\vec{t'}]) \subset \{y_1, \dots, y_n\}$*, then*
$KPi^+ \vdash \forall \vec{r}, \vec{s} \in Term_{Cl}.(r = s : A)^*[\vec{x}/(\vec{t}[\vec{x}/\vec{r}]); (\vec{t'}[\vec{x}/\vec{s}]), \vec{x}/\vec{r}; \vec{s}] \leftrightarrow$
$(r[\vec{x}/\vec{t}] = s[x/\vec{t'}] : A[x/\vec{t}])^*[\vec{x}/\vec{r}; \vec{s}]$.

Proof: (a): Induction on the definition of r g-term:
If r is a variable, the assertion is trivial, if $r = C(r_1, \dots, r_n)$ it follows by IH.
Let $r = \lambda y.s$. W.l.o.g. $x_i \neq y$, $x_i \in FV(r)$. Then, since the substitution is allowed, follows $y \notin FV(t_i)$. W.l.o.g. $y_i \neq y$. Then $r[\vec{x}/\vec{t}][\vec{y}/\vec{s}] = (\lambda y.(s[\vec{x}/\vec{t}]))[\vec{y}/\vec{s}] = \lambda y.(s[\vec{x}/\vec{t}][\vec{y}/\vec{s}]) = \lambda y.(s[\vec{x}/(\vec{t}[\vec{y}/\vec{s}]), \vec{y}/\vec{s}]) = r[\vec{x}/(\vec{t}[\vec{y}/\vec{s}]), \vec{y}/\vec{s}]$.
(b) Induction on the definition of A g-type:
$C = N, N_k, U$ is trivial.
$C = \Pi x \in A.B$. Let $[\vec{x'}/\vec{t'}] := [\vec{x}/\vec{t}] \setminus \{x\}$, $[\vec{y'}/\vec{r'}] := [\vec{y}/\vec{r}] \setminus \{x\}$, Then

$$
\begin{aligned}
&C[\vec{x}/\vec{t}]^{basis}[\vec{y}/\vec{r}] \\
&\quad \{pair(\lambda x.t, \lambda x'.t') \in Term_{Cl} \times Term_{Cl} | \forall pair(r, r') \in A[\vec{x}/\vec{t}]^*[\vec{y}/\vec{r}]. \\
&\qquad pair(t[x/r], t'[x'/r']) \in B[\vec{x'}/\vec{t'}]^*[x/r, \vec{y}/\vec{r}] \\
&\qquad\quad \wedge B[\vec{x'}/\vec{t'}]^*[x/r, \vec{y}/\vec{r}] B[\vec{x'}/\vec{t'}]^*[x/r', \vec{y}/\vec{r}]\}
\end{aligned}
$$

Now $x'_i \in FV(B) \to x \notin FV(t_i)$. Therefore

$$
\begin{aligned}
&C[\vec{x}/\vec{t}]^{basis}[\vec{y}/\vec{r}] \\
&\quad = \{pair(\lambda x.t, \lambda x'.t') \in Term_{Cl} \times Term_{Cl} | \forall pair(r, r') \in A^*[\vec{x}/\vec{t}[\vec{y}/\vec{r}], \vec{y}/\vec{r}]^*. \\
&\qquad pair(t[x/r], t'[x'/r']) \in B^*[\vec{x'}/(\vec{t'}[x/r, \vec{y}/\vec{r}]), x/r, \vec{y}/\vec{r}] \\
&\qquad\quad \wedge B^*[\vec{x'}/(\vec{t'}[x/r, \vec{y}/\vec{r}]), x/r, \vec{y}/\vec{r}] = B^*[\vec{x'}/(\vec{t'}[x/r, \vec{y}/\vec{r}]), x/r', \vec{y}/\vec{r}]\} \\
&\quad = \{pair(\lambda x.t, \lambda x'.t') \in Term_{Cl} \times Term_{Cl} | \forall pair(r, r') \in A^*[\vec{x}/\vec{t}[\vec{y}/\vec{r}], \vec{y}/\vec{r}]^*. \\
&\qquad pair(t[x/r], t'[x'/r']) \in B^*[x/r, \vec{x}/(\vec{t}[\vec{y}/\vec{r}]), \vec{y}/\vec{r}] \wedge \\
&\qquad\quad B^*[x/r, \vec{x}/(\vec{t}[\vec{y}/\vec{r}]), \vec{y}/\vec{r}] = B^*[x/r', \vec{x}/(\vec{t}[\vec{y}/\vec{r}]), \vec{y}/\vec{r}]\} \\
&\quad = C^{basis}[\vec{x}/(\vec{t}[\vec{y}/\vec{r}]), \vec{y}/\vec{r}].
\end{aligned}
$$

The cases $C = \Sigma x \in A.B, Wx \in A.B, A + B$ follow in a similar way.
Case $C = I(A, r, s)$: (a) and IH for A.
Case $C = T(t)$: (a).
(c) - (e) are immediate consequences of (a), (b).

Lemma 6.13 *For every g-type A $FV(A) \subset \{x_1, \dots, x_n\}$, follows*

(a) $\forall \vec{r}, r, r', s, s' \in Term_{Cl}.(r \rightarrow_{red} r') \rightarrow (s \rightarrow_{red} s')$
$\rightarrow pair(r, s) \in A^*[\vec{x}/\vec{r}] \rightarrow pair(r', s') \in A^*[\vec{x}/\vec{r}]$.

(b) $\forall \vec{r}, r, r' \in Term_{Cl}.pair(r, r') \in A^*[\vec{x}/\vec{r}] \rightarrow \exists s, s' \in Term_{nf}.r \rightarrow_{red} s \wedge r' \rightarrow_{red} s'$.

Proof: easy, since for each type, *Compl* was applied to some set.

Definition 6.14 (a) *Define*

$$Stable(a) := \forall r, s, r', s' \in Term_{Cl}.pair(r, s) \in a \rightarrow r =_\alpha r' \rightarrow s =_\alpha s' \rightarrow pair(r', s') \in a$$

(b) *For every g-type A with $FV(A) = \{x_1, \dots, x_n\}$ we define*

$$Flex(A) := \forall r_1, \dots, r_n, s_1, \dots, s_n \in Term_{Cl}.(r_1 =_\alpha s_1 \wedge \dots \wedge r_n =_\alpha s_n) \rightarrow A^*[\vec{x}/\vec{r}] = A^*[\vec{x}/\vec{s}]$$

Lemma 6.15 *For every g-type C, D with $FV(C) = \{x_1, \dots, x_n\}$ and $C =_\alpha D$ we have*

(a) $KPi^+ \vdash Flex(C)$

(b) $KPi^+ \vdash \forall r_1, \dots, r_n.Stable(C^*[\vec{x}/\vec{r}])$

(c) $KPi^+ \vdash \forall r_1, \dots, r_n.C^*[\vec{x}/\vec{r}] = D^*[\vec{x}/\vec{r}]$.

Proof of (a) - (c) by induction on definition of g-types, for (c) by side induction on $C =_\alpha D$. In (c) the case $C = D$ is trivial, and if $C =_\alpha C' =_\alpha D$, the assertion follows by IH.

We write $\vec{r} =_\alpha \vec{s}$ for $(r_1 =_\alpha s_1 \wedge \dots \wedge r_n =_\alpha s_n)$.

First note, that $Stable(a) \rightarrow Stable(Compl(a))$ by lemma 5.3 (d).

Case N, N_k: Easy.

Case $C = \Pi x \in A.B$: Since

$$\forall \vec{r}, \vec{s}.\vec{r} =_\alpha \vec{s} \rightarrow A[\vec{x}/\vec{r}] = A[\vec{x}/\vec{s}] \wedge \forall r \in Term_{Cl}.B[x/r, \vec{x}/\vec{r}] = B[x/r, \vec{x}/\vec{s}]$$

follows $Flex(\Pi x \in A.B)$, and if $\lambda x.t =_\alpha \lambda x''.t''$, $\lambda x'.t' =_\alpha \lambda x'''.t'''$,

$$pair(\lambda x.t, \lambda x'.t') \in C^{basis}[\vec{x}/\vec{r}],$$

then for $pair(r, r') \in A^*[\vec{x}/\vec{r}]$ follows by 5.2 (e) $t[x/r] =_\alpha t''[x''/r]$, $t'[x'/r'] =_\alpha t'''[x'''/r']$, $B^*[x/r, \vec{x}/\vec{r}] = B^*[x/r', \vec{x}/\vec{r}]$, and by $Stable(B^*[x/r, \vec{x}/\vec{r}]$, follows

$$pair(t''[x''/r], t'''[x'''/r']) \in B^*[x/r, \vec{x}/\vec{r}]$$

and

$$pair(\lambda x''.t'', \lambda x'''.t''') \in C^{basis}[\vec{x}/\vec{r}]$$

For (c), if $D = \Pi x \in A'.B'$ with $A =_\alpha A'$ $B =_\alpha B'$, the assertion follows by IH, if $D = \Pi y \in A.B[x/y]$, $y \notin FV(B)$, $B[x/y]$ allowed, follows assuming under the assumption $\vec{r} =_\alpha \vec{s}$

$$\forall r \in Term_{Cl}.B[x/y]^*[y/r, \vec{r}] = B^*[x/r, y/r, \vec{r}] = B^*[x/r, \vec{r}],$$

and therefore follows $C^*[\vec{r}] = D^*[\vec{r}]$.

Case $C = \Sigma x \in A.B$: Stability follows immediately by IH, and if $r =_\alpha r''$, $s =_\alpha s''$, $r' =_\alpha r'''$, $s' =_\alpha s'''$, $pair(p(r, s), p(r', s')) \in C^{basis}[\vec{x}/\vec{r}]$, $pair(r'', r''') \in A^*[\vec{x}/\vec{r}]$, follows by IH $B[x/r, \vec{x}/\vec{r}] = B[x/r'', \vec{x}/\vec{r}]$, and $B[x/r', \vec{x}/\vec{r}] = B[x/r''', \vec{x}/\vec{r}]$. therefore by stability

$$pair(s', s'') \in B[x/r, \vec{x}/\vec{r}] = B[x/r'', \vec{x}/\vec{r}]$$

(c) follows as for $\Pi x \in A.B$.

Case $C = Wx \in A.B$: As in the Σ-case follows using the IH, $F(\vec{r}, u) = F(\vec{s}, u)$. Further we see, that $Stable(u) \to Stable(F^{basis}(\vec{r}, u))$.

(c) follows as for $\Pi x \in A.B$.

Case $C = (A + B)$: as before, (c) is easy.

Case $C = I(A, r, s)$: $Stable(I^{basis}(A, r, s))$ is trivial. $Flex(I(A, r, s))$: If $\vec{r}{=_\alpha}\vec{s}$, then $s[\vec{x}/\vec{r}]{=_\alpha}s[\vec{x}/\vec{s}]$, $t[\vec{x}/\vec{r}]{=_\alpha}t[\vec{x}/\vec{s}]$, $A^*[\vec{x}/\vec{r}] = A^*[\vec{x}/\vec{s}]$, and therefore $pair(s[\vec{x}/\vec{r}], t[\vec{x}/\vec{r}]) \in A[\vec{x}/\vec{r}] \leftrightarrow pair(s[\vec{x}/\vec{s}], t[\vec{x}/\vec{s}]) \in A[\vec{x}/\vec{s}]$.

For (c), $D = I(B, r', s')$, with $r{=_\alpha}r'$, $s{=_\alpha}s'$, $A{=_\alpha}B$, $A^*[\vec{r}] = B^*[\vec{r}]$, by 5.2 (d) $r[\vec{r}]{=_\alpha}r'[\vec{r}]$, $s[\vec{r}]{=_\alpha}s'[\vec{r}]$ and, using $Stable(A^*[\vec{r}])$, follows $pair(r[\vec{r}], s[\vec{r}]) \in A^*[\vec{r}] \leftrightarrow pair(r'[\vec{r}], s'[\vec{r}]) \in A^*[\vec{r}] \leftrightarrow pair(r'[\vec{r}], s'[\vec{r}]) \in B^*[\vec{r}]$.

Cases $C = U, T(t)$: We define

$$\begin{aligned} Stable_U(u) :=\quad & \forall s, s', t, t' \in Term_{Cl}.\forall b \in TC(u). \\ & s{=_\alpha}s' \to t{=_\alpha}t' \to pair(s, b, t) \in u \to (pair(s', b, t') \in u \wedge Stable(b)) \end{aligned}$$

We have $Stable_U(u) \to Stable_U(Compl_U(u))$. We have $Stable(u_{nat}^{basis})$, $Stable(u_{fin}^{basis}(k))$,

$(\forall a, a', a'', a'''.(pair(a, a') \in b \wedge a{=_\alpha}a'' \wedge a'{=_\alpha}a''') \to (f(a) = f(a'') \wedge f(a') = f(a''') \wedge Stable(f(a)))) \to Stable(b) \to (Stable(u_\pi^{basis}(b, f)) \wedge Stable(u_\sigma^{basis}(b, f)) \wedge (Stable(v) \to Stable(u_w^{basis}(b, f, v))))$,

$(Stable(b) \wedge Stable(c)) \to Stable(u_+^{basis}(b, c))$,

$Stable(u_i(b, r, s))$.

Further $Stable(b) \to r{=_\alpha}r' \to s{=_\alpha}s' \to u_i^{basis}(b, r, s) = u_i^{basis}(b, r', s')$.

$(Stable_U(u) \wedge r{=_\alpha}r' \wedge s{=_\alpha}s' \wedge \lambda x.t{=_\alpha}\lambda x'.t' \wedge \lambda y.v{=_\alpha}\lambda y'.v')) \to \phi(r, x, t, s, y, v, b, f, u) \to \phi(r', x', t', s', y', v', b, f, u)$.

$(Stable_U(u) \wedge r{=_\alpha}r' \wedge s{=_\alpha}s \wedge t{=_\alpha}t' \wedge v{=_\alpha}v') \to \psi_+(r, s, t, v, b, c, u) \to \psi(r', s', t', v', u)$.

$(Stable_U(u) \wedge r{=_\alpha}r' \wedge s{=_\alpha}s' \wedge t{=_\alpha}t'\tilde{r}{=_\alpha}\tilde{r}' \wedge \tilde{s}{=_\alpha}\tilde{s}' \wedge \tilde{t}{=_\alpha}\tilde{t}') \to \psi_i(r, s, t, \tilde{r}, \tilde{s}, \tilde{t}, b, c, u) \to \psi_i(r', s', t', \tilde{r}', \tilde{s}', \tilde{t}', b, c, u)$

From all this together we conclude $Stable_U(u) \to Stable_U(\tilde{U}(u))$ and therefore $Stable_U(\hat{U})$. Now immediately follows $Stable(U^*)$, $Flex(U)$ is trivial, $Stable(T(t)^*[\vec{x}/\vec{r}])$, and since for $\vec{r}{=_\alpha}\vec{s}$, $t[\vec{x}/\vec{r}]{=_\alpha}t[\vec{x}/\vec{s}]$, follows

$$pair(t[\vec{x}/\vec{r}], b, t[\vec{x}/\vec{r}]) \in \hat{U} \leftrightarrow pair(t[\vec{x}/\vec{s}], b, t[\vec{x}/\vec{s}]) \in \hat{U},$$

$$(T(t))^*[\vec{x}/\vec{r}]{=_\alpha}(T(t))^*[\vec{x}/\vec{s}]$$

Now, if $T(t){=_\alpha}D$, $D = T(t')$ with $t{=_\alpha}t'$, $T(t)^*[\vec{r}] = T(x)^*[x/t[\vec{r}]] = T(x)^*[x/t'[\vec{r}]] = T(t')^*[\vec{r}]$.

To state our Main Lemma, we need to express, that, if we assume elements of the types of the context, the interpretation of the conclusion Θ of a statement of Martin-Löf is valid. Since we need, that this is independent of the choice of equal elements of A_i, we will introduce the following abbreviation:

Definition 6.16 *Let $\Gamma \equiv x_1 : A_1, \ldots, x_k : A_k$ be a g-context.*

$$\begin{aligned} \forall\Gamma^=(\vec{r}; \vec{s}).\phi :\equiv\ & \forall r_1, \ldots, r_k, s_1, \ldots, s_k \in Term_{Cl}.(pair(r_1, s_1) \in A_1^*[\,] \wedge \\ & pair(r_2, s_2) \in A_2^*[x_1/r_1] \wedge \cdots \wedge pair(r_k, s_k) \in A_k^*[x_1/r_1, \ldots, x_{k-1}/r_{k-1}]) \to \phi \end{aligned}$$

"Assume $\Gamma^=(\vec{r}; \vec{s})$" means:

"Assume $r_1, \ldots, r_k, s_1, \ldots, s_k \in Term_{Cl}$ such that $pair(r_1, s_1) \in A_1^*[\,] \wedge$
$pair(r_2, s_2) \in A_2^*[x_1/r_1] \wedge \cdots \wedge pair(r_k, s_k) \in A_k^*[x_1/r_1, \ldots, x_{k-1}/r_{k-1}]$.

Remark 6.17 *All the proofs, carried out before, can be carried out in KPi^+.*

Now we can state our Main Lemma. We need to prove that

$$ML_1^e W_T \vdash t : A \;\Rightarrow\; KPi^+ \vdash (t : A)^*.$$

But to carry out the proof, we need an assertion for each judgement of $ML_1^e W_T$, and further, that it respects equality:

Lemma 6.18 *(***Main lemma***)*

Let Γ, Δ be g-context-pieces, $x, x_i \in Var_{ML}$, A_i, A, B g-types, t, t' g-terms, θ a g-judgement. Assume $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.

(a) If $ML_1^e W_T \vdash \Gamma \Rightarrow t : A$, then

(i) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(t : A)^*[\vec{x}/\vec{r}; \vec{s}]$.

(ii) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A \ type)^*[\vec{x}/\vec{r}; \vec{s}]$.

(b) If $ML_1^e W_T \vdash \Gamma \Rightarrow t = t' : A$, then

(i) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(t = t' : A)^*[\vec{x}/\vec{r}; \vec{s}]$.

(ii) $KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A \ type)^*[\vec{x}/\vec{r}; \vec{s}]$.

(c) If $ML_1^e W_T \vdash \Gamma \Rightarrow A = A'$, then

$$KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A = A')^*[\vec{x}/\vec{r}; \vec{s}].$$

(d) If $ML_1^e W_T \vdash \Gamma \Rightarrow A$ type, then

$$KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A \ type)^*[\vec{x}/\vec{r}; \vec{s}].$$

(e) If $ML_1^e W_T \vdash \Gamma, x : A, \Delta \Rightarrow \theta$, then

$$KPi^+ \vdash \forall \Gamma^=(\vec{r}; \vec{s}).(A \ type)^*[\vec{x}/\vec{r}; \vec{s}].$$

Proof of the Main Lemma:
We proof simultaneously (a) - (e) by induction on the derivation. We write IH 3 for the Induction-hypothesis for the 3rd premise, etc. IH 3(d) for the Induction-hypothesis (d) for the 3rd premise of the rule etc.
If there is more than one rule of one category (as in the case of $(REFL)$), we refer to them by $(REFL)_1$, $(REFL)_2$, etc.
Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, $\Gamma' = y_1 : B_1, \ldots, y_m : B_m$.
If $\vec{r} = r_1, \ldots, r_n$, $i \leq n$, then $\hat{r}_i := r_1, \ldots, r_i$.
If $\theta = t : A$ or $t = t' : A$ or A type or $A = B$, let $\theta' = A$ type (the judgement treated in the cases (i) of (a) - (c), or which follows from the assertion in (d), (e).
Distinction by the last rule applied.
Case (ASS): Assume $\Gamma^=(\vec{r}; \vec{s})$, $pair(r, r') \in A^*[\vec{x}/\vec{r}]$. Then $x[\vec{x}/\vec{r}, x/r] = r$, $x[\vec{x}/\vec{s}, x/r'] = r'$, $(x : A)^*[\vec{x}/\vec{r}; \vec{s}, x/r; r']$

ad (a,ii): Direct by IH.

ad (e): if „$y : B$" part of Γ, by IH (e).

if $y : B \;\equiv x : A$ by IH (a,ii).

Case $(THIN)$: Assume $\Gamma^=(\vec{r}; \vec{s})$, $pair(r, r') \in A^*[\vec{n}]$, $pair(r'_i, s'_i) \in B_i^*[\vec{x}/\vec{r}, x/r, \hat{y}_i/\hat{r}_i]$.

Then $pair(r_i', s_i') \in B_i^*[\vec{x}/\vec{r}, \hat{y}_i/\hat{r}_i]$, by IH $\theta^*[\vec{x}/\vec{r}; \vec{s}, \vec{y}/\vec{r}'; \vec{s}']$, therefore $\theta^*[\vec{x}/\vec{r}; \vec{s}, x/r; r', \vec{y}/\vec{r}'; \vec{s}']$. and $(\theta')^*[\vec{x}/\vec{r}; \vec{s}, x/r; r', \vec{y}/\vec{r}'; \vec{s}']$.
(e): If „$y : B$" $\equiv x : A$, follows (e) from IH 1(d), otherwise as before by IH.

Case $(REFL)_1 - (REFL)_5$: Direct by IH.

Case $(SYM)_1$ Assume $\Gamma^{=}(\vec{r}; \vec{s})$. From $pair(r_i, s_i) \in A_i^*[\hat{r}_i]$ follows $pair(s_i, r_i) \in A_i^*[\hat{r}_i]$ and by IH (b,ii) $pair(s_i, r_i) \in A_i^*[\hat{s}_i]$. By IH (b,i) follows $pair(t[\vec{s}], t'[\vec{r}]) \in A^*[\vec{s}]$, and by IH (b,ii) $A^*[\vec{r}] = A^*[\vec{s}]$, and by 6.10 follows $(t' = t : A)^*[\vec{r}; \vec{s}]$.
(b,ii) follows from IH (b,ii).

Case $(SYM)_2$ Assume $\Gamma^{=}(\vec{r}; \vec{s})$. As for $(SYM)_1$ we have $pair(s_i, r_i) \in A_i^*[\hat{s}_i]$, by IH $A^*[\vec{s}] = B^*[\vec{r}]$ and therefore the assertion.

Case $(TRANS)_1$ Assume $\Gamma^{=}(\vec{r}; \vec{s})$. From $pair(r_i, s_i) \in A_i^*[\hat{r}_i]$ follows $pair(r_i, r_i) \in A_i^*[\hat{r}_i]$, $pair(t[\vec{r}], t'[\vec{r}]) \in A^*[\vec{r}]$, $pair(t'[\vec{r}], t''[\vec{s}]) \in A^*[\vec{r}]$ and therefore from $equiv(A^*[\vec{n}])$ follows $(t = t'' : A)^*[\vec{r}; \vec{s}]$. (b,ii) follows by IH.

Case $(TRANS)_2$ Assume $\Gamma^{=}(\vec{r}; \vec{s})$. From $pair(r_i, s_i) \in A_i^*[\hat{r}_i]$ follows $pair(r_i, r_i) \in A_i^*[\hat{r}_i]$, $A^*[\vec{r}] = B^*[\vec{r}]$, $B^*[\vec{r}] = C^*[\vec{s}]$ and therefore the assertion.

Case (SUB) Assume $\Gamma^{=}(\vec{r}; \vec{r}')$, $pair(s_i, s_i') \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Now by lemma 6.12

$$B_i[x/t]^*[\vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}, \hat{s}_i], \vec{r}, \hat{s}_i] = B_i^*[x/t[\vec{r}], \vec{r}, \hat{s}_i]$$

By IH 2 (a,i) $pair(t[\vec{r}], t[\vec{s}]) \in A^*[\vec{r}]$, therefore

$$\theta^*[\vec{x}/\vec{r}; \vec{r}', x/t[\vec{r}]; t'[\vec{r}'], \vec{y}/\vec{s}; \vec{s}'],$$

and by lemma 6.12 $\theta[x/t]^*[\vec{x}/\vec{r}; \vec{r}', \vec{s}]$, similarly for θ'.
Proof for (e): If „$y : B$" in Γ, follows assertion by IH.
If „$y : B$" in $\Gamma'[x/t]$, follows by IH $B^*[\vec{r}, x/\vec{t}[\vec{r}], \hat{s}_i] = B^*[\vec{r}', x/t[\vec{r}'], \hat{s}_i']$, and by 6.12 the assertion.

Case $(REPL1)$ Assume $\Gamma^{=}(\vec{r}; \vec{r}')$, $pair(s_i, s_i') \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. $pair(t[\vec{r}], t'[\vec{r}']) \in A^*[\vec{r}]$. By 6.12 follows $B_i^*[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i] = B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Therefore we have $pair(s_i, s_i') \in B_i^*[\vec{x}/\vec{r}, x/t[\vec{r}], \hat{y}_i/\hat{s}_i]$ Then by IH 1 $B^*[\vec{x}/\vec{r}, x/t[\vec{r}], \vec{y}/\vec{s}] = B^*[\vec{x}/\vec{r}', x/t'[\vec{r}'], \vec{y}/\vec{s}']$, and by 6.12 follows the assertion.
Proof for (e): From IH 2 follows as in $(REFL)$ the assertion for $\Gamma \Rightarrow t' = t : A$ and further as in $(TRANS)$ the assertion for $\Gamma \Rightarrow t = t : A$, which is the same as for $\Gamma \Rightarrow t : A$ and now the proof follows as in (SUB).

Case $(REPL2)$ Assume $\Gamma^{=}(\vec{r}, \vec{r}')$, $pair(s_i, s_i') \in B_i[x/t]^*[\vec{r}, \hat{s}_i]$. Then $pair(t[\vec{r}], t'[\vec{r}']) \in A^*[\vec{r}]$, and by IH 1(a,i)

$$(s = s : B)^*[\vec{x}/\vec{r}; \vec{r}', x/t[\vec{r}]; t'[\vec{r}'], \vec{y}/\vec{s}; \vec{s}']$$

and by the 6.12 follows the assertion for (b,i). (b,ii) follows as in $(REPL1)$, using that we have the assertion for $\Gamma \Rightarrow t = t : A$, and (e) follows exactly as in $(REPL1)$.

Case $(REPL3)$ Assume $\Gamma^{=}(\vec{r}; \vec{s})$. We have $(t = t : A)[\vec{r}; \vec{s}]$, and, since we have $pair(r_i, r_i) \in A_i^*[\hat{r}_i]$, $A^*[\vec{r}] = B^*[\vec{r}]$, therefore the assertion for the first rule. The assertion for the second rule is similar. Using the proofs of $(REFL)$ and $(TRANS)$ follows the assertion for $\Gamma \Rightarrow B = B$ and therefore the assertion for (b,ii).

Case $(ALPHA)$: Immediate by the IH since if $A =_\alpha A'$, $t =_\alpha t'$, $A[\vec{s}] = A'[\vec{s}]$, $t[\vec{r}] =_\alpha t'[\vec{r}]$ and $pair(t[\vec{r}], t[\vec{r}]) \in A^*[\vec{r}] \leftrightarrow pair(t[\vec{r}], t'[\vec{r}]) \in A^*[\vec{r}]$.
Case (N_k^T), (N^T) Nothing to prove.

Case $(\Pi^{T,=})$ Assume $\Gamma^{=}(\vec{r};\vec{s})$. By IH (e) $A^*[\vec{r}] = A'^*[\vec{s}]$, and, if $pair(r,s) \in A^*[\vec{r}]$, follows $pair(r,r), pair(s,s) \in A^*[\vec{r}]$, therefore by IH $B^*[\vec{r}, x/r] = B'^*[\vec{r}, x/r]$, $B^*[\vec{s}, x/s] = B'^*[\vec{r}, x/s]$, $\Pi x \in A.B^*[\vec{r}] = (\Pi x \in A'.B')^*[\vec{s}]$.

Cases $(\Sigma^{T,=})$, $(W^{T,=})$, $(+^{T,=})$: similarly.

Case $(I^{T,=})$: Assume $\Gamma^{=}(\vec{r};\vec{s})$. By IH (1) we have

$$pair(t[\vec{r}], t'[\vec{s}]) \in A^*[\vec{r}],\ pair(s[\vec{r}], s'[\vec{s}]) \in A^*[\vec{r}])$$

and by IH 1,(b,ii) $A^*[\vec{r}] = A'^*[\vec{s}]$. Therefore we have $pair(t[\vec{r}], s[\vec{r}]) \in A^*[\vec{r}] \Leftrightarrow (t'[\vec{s}], s'[\vec{s}]) \in A'^*[\vec{s}]$.

Cases (N^I_k), $(N^{I,=})_1$ Nothing to prove.

Case $(N^{I,=})_2$: Assume $\Gamma^{=}(\vec{r};\vec{s})$. By IH we have for some $k \in \mathbb{N}$ $t[\vec{r}] \longrightarrow_{red} S^k 0 \wedge t'[\vec{s}] \longrightarrow_{red} S^k 0$, therefore $St[\vec{r}] \longrightarrow_{red} S^{k+1}0$, $St'[\vec{s}] \longrightarrow_{red} S^{k+1}0$, and we have the assertion.

Case $(\Pi^{I,=})$: Assume $\Gamma^{=}(\vec{r};\vec{s})$, $pair(r,s) \in A^*[\vec{r}]$. Then by IH (b,i) $pair(t[x/r,\vec{r}], t'[x/s,\vec{s}]) \in B^*[x/r,\vec{r}] = B^*[\vec{r}, x/r]$, $pair((\lambda x.t)[\vec{x}/\vec{r}], (\lambda x.t')[\vec{x}/\vec{s}]) \in (\Pi x \in A.B)^*[\vec{r}]$.

(b,ii) follows as in $(\Pi^{T,=}_1)$, since from IH (b,ii) follows (d) for $x : A \Rightarrow B$ type.

Case $(\Sigma^{I,=})$: Assume $\Gamma^{=}(\vec{r};\vec{s})$. By IH $pair(s[\vec{r}], s'[\vec{s}]) \in A^*[\vec{r}]$, further $pair(t[\vec{r}], t'[\vec{s}]) \in (B[x/s])^*[\vec{r}] = B^*[x/s[\vec{r}], \vec{r}] = B^*[x/s'[\vec{s}], \vec{s}]$. Therefore $pair(p(s,t)[\vec{r}], p(s',t')[\vec{s}]) \in (\Sigma x \in A.B)^*[\vec{n}]$.

(b,ii) follows as in $(\Sigma^{T,=})$.

Case $(W^{I,=})$: Let $\alpha_{I,n}, F$ be as in the definition of $Wx \in A.B^*$, $\alpha := \alpha_{I,n}$. Assume $\Gamma^{=}(\vec{r};\vec{s})$. Then by IH $pair(r[\vec{r}], r'[\vec{s}]) \in A^*[\vec{r}]$, $s[\vec{r}] \longrightarrow_{red} \lambda x.t$, $s'[\vec{s}] \longrightarrow_{red} \lambda x'.t'$, $B^*[x/r[\vec{r}], \vec{r}] = B^*[x/r'[\vec{s}], \vec{s}]$, and

$$\forall pair(u,u') \in B[x/t]^*[\vec{x}/\vec{r}](= B^*[x/t[\vec{r}], \vec{x}/\vec{r}])\exists \gamma < \alpha.pair(t[x/u], t'[x/u']) \in F(\vec{r}, \gamma)$$

By $(\Delta_0 - coll)$ and $Ad(L_\alpha)$ there exist a $\delta < \alpha$ such that the γ can be chosen to be $< \delta$. Then $pair(sup(r,s)[\vec{r}], sup(r,s)[\vec{s}] \in F(\vec{r}, \gamma+1) \subset Wx \in A.B^*[\vec{r}]$.

(b,ii) follows as in $(W^{T,=})$.

Case $(+^{I,=})$: easy.

Case $(I^{I,=})$: obvious.

Case $(N^{E,=}_k)$: Assume $\Gamma^{=}(\vec{r};\vec{s})$. By I.H. 1 follows $t[\vec{r}] \longrightarrow_{red} n_k$, $t'[\vec{s}] \longrightarrow_{red} n_k$, for some $n < k$, $s_i[\vec{r}] \longrightarrow_{red} \tilde{s}_i \in Term_{nf}$, $s'_i[\vec{s}] \longrightarrow_{red} \tilde{s}'_i \in Term_{nf}$ for some $pair(\tilde{s}_i, \tilde{s}'_i) \in A[x/i_k]^*[\vec{r}] = A^*[x/i_k, \vec{r}]$.

Therefore by lemma 5.3 (c)

$$C_k(t, s_0, \ldots, s_{k-1})[\vec{r}] \longrightarrow_{red} C_k(n_k, \tilde{s}_0, \ldots, \tilde{s}_{k-1}) \longrightarrow_{red} \tilde{s}_n,$$

$$C_k(t', s_0, \ldots, s_{k-1})[\vec{s}] \longrightarrow_{red} \tilde{s}'.$$

$t[\vec{r}] \longrightarrow_{red} n_k$, therefore $pair(t[\vec{r}], n_k) \in N_k$. By $pair(r_i, r_i) \in A_i[\hat{r}_i]$ and IH for the last premise follows $A[x/t]^*[\vec{r}] = A^*[x/t^*[\vec{r}], \vec{r}] = A^*[x/n_k, \vec{r}]$ and we are done.

(b,ii) follows, since from IH 1 follows as in (SYM), $(TRANS)$ the assertion for $t = t : N_k$, as in (SUB).

(b,ii) follows as in (SUB).

Case$(N^{E,=})$: Assume $\Gamma^{=}(\vec{r};\vec{s})$. Then by IH 1 $pair(t,t')[\vec{r}] \in N$, therefore $t[\vec{r}] \longrightarrow_{red} S^n 0$, $t'[\vec{r}] \longrightarrow_{red} S^n 0$ for some $n < \omega$. Further by IH 2 and 6.13 (b) exist $\tilde{s}_0, \tilde{s}'_0 \in Term_{nf}$ such that $s_0[\vec{r}] \longrightarrow_{red} \tilde{s}_0$, $s_1[\vec{r}] \longrightarrow_{red} \tilde{s}'_0$, $pair(\tilde{s}_0, \tilde{s}'_0) \in A[x/0]^*[\vec{r}] = A^*[x/u, \vec{r}]$.

Let $[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x, y\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{s}] \setminus \{x, y\}$.

$$P(t, s_0, (x,y)s_1)[\vec{r}] \longrightarrow_{red} P(S^n 0, \widetilde{s}_0, \lambda x.\lambda y.(s_1[\vec{x}'/\vec{r}'])),$$
$$P(t', s_0', (x,y)s_1')[\vec{s}] \longrightarrow_{red} P(S^n 0, \widetilde{s}_0', \lambda x.\lambda y.(s_1'[\vec{x}'/\vec{r}'])).$$

Let $P_0(r) := P(r, \widetilde{s}_0, \lambda x.\lambda y.(s_1[\vec{x}'/\vec{r}']))$, $P_1(r) := P(r, \widetilde{s}_0', \lambda x.\lambda y.(s_1'[\vec{x}'/\vec{s}']))$. We show: $\forall m \in \mathbb{N}.pair(P_0(S^m 0), P_1(S^m 0)) \in A^*[z/S^m 0, \vec{r}]$. We will now conclude as in the case $N_k^{E,=}$, $A[z/t]^*[\vec{r}] = A^*[z/t[\vec{r}], \vec{r}] = A^*[z/S^n 0, \vec{r}]$, and have now assertion (b,i).
If $m = 0$, $P_0(S^m 0) \longrightarrow_{red} \widetilde{s}_0$, $P_1(S^m 0) \longrightarrow_{red} \widetilde{s}_0'$, $pair(\widetilde{s}_0, \widetilde{s}_0') \in A^*[z/0, \vec{r}]$.
If $m = k+1$, follows by IH $P_0(S^k 0) \longrightarrow_{red} \widetilde{s}$, $P_1(S^k 0) \longrightarrow_{red} \widetilde{s}'$, $\widetilde{s}, \widetilde{s}' \in Term_{nf}$, $pair(\widetilde{s}, \widetilde{s}') \in A^*[z/S^k 0, \vec{r}] = A[z/x]^*[x/S^k 0, \vec{r}]$.

$$\begin{aligned} P_0(S^m 0) &\longrightarrow_{red} (\lambda x.\lambda y.(s_1[\vec{r}']))S^k 0 P_0(S^k 0) \longrightarrow_{red} (\lambda y.s_1[x/S^k 0, \vec{r}'])\widetilde{s} \\ &\longrightarrow_{red} s_1[\vec{x}'/\vec{r}', x/S^k 0, y/\widetilde{s}] \\ P_1(S^m 0) &\longrightarrow_{red} s_1'[\vec{x}'/\vec{s}', x/S^k 0, y/\widetilde{s}']. \end{aligned}$$

Now $pair(S^k 0, S^k 0) \in N^*$, therefore by IH 3 follows

$$pair(s_1[\vec{x}'/\vec{r}', x/S^k 0, y/\widetilde{s}], s_1'[\vec{x}'/\vec{s}', x/S^k 0, y/\widetilde{s}']) \in A[z/Sx]^*[\vec{r}] = A^*[z/S^m 0, \vec{r}],$$

and the side induction is finished.
(b,ii) follows as in the case $N_k^{E,=}$.
Case($\Pi^{E,=}$): Assume $\Gamma^=(\vec{r}; \vec{s})$. By IH 1,2 there exist $\widetilde{t}_1, \widetilde{t}_1' \in Term_{nf}$ such that $t_1[\vec{r}] \longrightarrow_{red} \widetilde{t}_1$, $t_1'[\vec{s}] \longrightarrow_{red} \widetilde{t}_1'$ $pair(\widetilde{t}_1, \widetilde{t}') \in A^*[\vec{r}]$, and there are $r, r' \in Term$ and Variables $x, x' \in Var_{ML}$ such that

$$t_0[\vec{r}] \longrightarrow_{red} \lambda x.r, \; t_0'[\vec{r}'] \longrightarrow_{red} \lambda x'.r', \; pair(\lambda x.r, \lambda x'.r') \in (\Pi x \in A.B)^{basis}[\vec{r}].$$

Therefore

$$Ap(t_0, t_1)[\vec{r}] \longrightarrow_{red} Ap(\lambda x.r, \widetilde{t}_1) \longrightarrow_{red} r[x/\widetilde{t}_1, \vec{r}],$$
$$Ap(t_0', t_1')[\vec{s}] \longrightarrow_{red} r'[x'/\widetilde{t}_1, \vec{s}]$$
$$pair(r[x/\widetilde{t}_1, \vec{r}], r'[x'/\widetilde{t}_1', \vec{s}]) \in B^*[x/\widetilde{t}_1, \vec{r}].$$

As before we conclude

$$pair(t_1[\vec{r}], t_1[\vec{r}]) \in A^*[\vec{r}]$$
$$pair(\widetilde{t}_1, t_1[\vec{r}]) \in A^*[\vec{r}]$$
$$B^*[x/\widetilde{t}_1, \vec{r}] = B^*[x/t_1[\vec{r}], \vec{r}] = B[x/t_1]^*[\vec{r}],$$

and we have IH (b,i).
(b,ii) follows as in the case ($N_k^{E,=}$).
Case($\Sigma^{E,=}$): Assume $\Gamma^=(\vec{r}; \vec{s})$. By IH 1 exist $s, s', t, t' \in Term_{nf}$ such that

$$r[\vec{r}] \longrightarrow_{red} p(s,t), \; r'[\vec{s}] \longrightarrow_{red} p(s', t'), \; pair(s, s') \in A^*[\vec{r}], \; pair(t, t') \in B^*[x/s, \vec{r}].$$

Then $p_0(r[\vec{r}]) \longrightarrow_{red} s$, $p_0(r'[\vec{s}]) \longrightarrow_{red} s'$, and we are done for the first rule, and $p_1(r[\vec{r}]) \longrightarrow_{red} t$, $p_1(r'[\vec{s}]) \longrightarrow_{red} t'$, and since from $pair(s, s') \in A[\vec{r}]$, follows

$$pair(s, s') \in A[\vec{r}], \; pair(p_0(r)[\vec{r}], s) \in A[\vec{r}],$$

therefore by IH 2

$$B^*[x/s,\vec{r}] = B^*[x/p_0(r)[\vec{r}],\vec{r}] = B[x/p_0(r)]^*[\vec{r}]$$

follows (b,i) for the second rule.
(b,ii) is in $(\Sigma^{E,=})_1$ trivial, in $(\Sigma^{E,=})_2$ we use the proof of $(\Sigma^{E,=})_1$ and argue as before.
Case($W^{E,=}$): Assume $\Gamma^=(\vec{r};\vec{s})$, $\alpha := \alpha_{I,n}$ as in the definition of $(Wx \in A.B)^*$.
By IH $t_0[\vec{r}] \rightarrow_{red} \widetilde{t}_0$, $t'_0[\vec{s}] \rightarrow_{red} \widetilde{t}'_0$, $pair(\widetilde{t}_0, \widetilde{t}'_0) \in F^\delta(\vec{r},\cdot)(\emptyset)$. for some $\delta < \alpha$. Let

$$[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x,y,z\},$$

$$[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{s}] \setminus \{x,y,z\},$$

$$R_0(r) := R(r,(x,y,z)t_2)[\vec{r}](= R(r,\lambda x.\lambda y.\lambda z.(t_2[\vec{r}'])))$$

$$R_1(r) := R(r,(x,y,z)t'_2)[\vec{s}].$$

We show by induction on γ,

$$(+) \qquad \forall \gamma < \alpha.\forall pair(\widetilde{s},\widetilde{s}') \in F^\gamma(\vec{r},\cdot)(\emptyset).pair(R_0(\widetilde{s}),R_1(\widetilde{s}')) \in C^*[u/\widetilde{s},\vec{r}]$$

Since $C[u/t_0]^*[\vec{r}] = C^*[u/t_0[\vec{r}],\vec{r}] = C^*[u/\widetilde{t}_0,\vec{r}] = C^*[u/\widetilde{t}'_0,\vec{s}]$ (using arguments as before), follows the assertion.
The case $\gamma = 0$ is trivial, and if $\gamma \in Lim$ follows the assertion by IH
Let now
$\gamma = \gamma' + 1$, $u' := F^{\gamma'}(\vec{r},\cdot)(\emptyset)$, $pair(\widetilde{s},\widetilde{s}') \in F(\vec{r},u')$.
If $\widetilde{s} \rightarrow_{red} s$, $\widetilde{s}' \rightarrow_{red} s'$, $pair(s,s') \in F^{basis}(\vec{s},\cdot)(\emptyset)$, $pair(R_0(s),R_1(s')) \in C^*[u/s,\vec{r}]$, follows $pair(R_0(\widetilde{s}),R_1(\widetilde{s}')) \in C^*[u/s,\vec{r}]$, further, like similar arguments before, $C^*[u/s,\vec{r}] = C^*[u/\widetilde{s},\vec{r}] = C^*[u/\widetilde{s}']$. We therefore assume $pair(\widetilde{s},\widetilde{s}') \in F^{basis}(\vec{r},u')$.
Let $pair(\widetilde{s},\widetilde{s}') = pair(sup(r,\lambda x.s), sup(r',\lambda x'.s'))$, $pair(r,r') \in A^*[\vec{r}]$. Let $pair(r'',r''') \in B^*[x/r,\vec{r}]$. Then $r'' \rightarrow_{red} \widetilde{r}$, $r''' \rightarrow_{red} \widetilde{r}'$ for $pair(\widetilde{r},\widetilde{r}') \in B^*[x/r,\vec{r}]$, $\widetilde{r},\widetilde{r}' \in Term_{nf}$, and we have $pair(s[x/r''], s'[x'/r''']) \in u'$ and

$$(*) \qquad pair(s[x/\widetilde{r}], s'[x'/\widetilde{r}']) \in u'$$

Since $u' \subset (Wx \in A.B)^*[\vec{r}]$ follows from the first of these assertions

$$pair(\lambda x.s,\lambda x'.s') \in (B \rightarrow Wx \in A.B)^*[\vec{r}]$$

Further, for $pair(\widetilde{r},\widetilde{r}') \in B^*[x/r,\vec{r}]$,

$$(R_0((\lambda x.s)v))[v/r''] \rightarrow_{red} R_0(s[x/\widetilde{r}])(v \notin FV(\lambda x.s))$$

$$(R_1((\lambda x.s')v'))[v'/r'''] \rightarrow_{red} R_1(s'[x/\widetilde{r}'])(v' \notin FV(\lambda x.s))$$

and by side IH, follows

$$\begin{aligned} pair((R_0((\lambda x.s)v))[v/r''], (R_1((\lambda x'.s')v))[v'/r''']) &\in C^*[u/(s[x/\widetilde{r}],\vec{r}]) \\ &= C^*[u/(s[x/r'']),\vec{r}]) \end{aligned}$$

Now we have $pair(r_i,r_i) \in A_i[\hat{r}_i]$, $Ap(\lambda x.s,r') \rightarrow_{red} s[x/\widetilde{r}]$, and by $(*)$, $u' \subset (Wx \in A.B)^*[\vec{r}]$, $equiv((Wx \in A.B)^*[\vec{r}])$ and 6.13 follows

$$pair(s[x/\widetilde{r}], Ap(\lambda x.s,\widetilde{r})) \in (Wx \in A.B)^*[\vec{r}]$$

therefore

$$C[u/Ap(y,v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C^*[u/Ap(\lambda x.s, r''), \vec{r}] = C^*[u/(s[x/\tilde{r}]), \vec{r}]$$

further

$$C[u/Ap(y,v)]^*[v/r'', y/\lambda x.s, \vec{x}/\vec{r}] = C[u/Ap(y,v)]^*[v/r''', y/\lambda x.s, \vec{x}/\vec{r}],$$

and we have

$$pair(\lambda v.R_0((\lambda x.s)v), \lambda v'.R_1((\lambda x'.s')v')) \in (\Pi v \in B.C[u/Ap(y,v)])^*[y/\lambda x.s, \vec{x}/\vec{r}]$$

Now by IH 2 follows

$$pair(t_2[x/r, y/\lambda x.s, z/\lambda v.R_0((\lambda x.s)v), \vec{r}], t_2'[x/r', y/\lambda x'.s', z/\lambda v'.R_1((\lambda x.s')v'), \vec{s}]) \\ \in C[u/sup(x,y)]^*[x/r, y/\lambda x.s, \vec{r}]$$

Since

$$C[u/sup(x,y)]^*[x/r, y/\lambda x.s, \vec{r}] = C^*[u/sup(r, \lambda x.s), \vec{r}] = C^*[u/s, s],$$

and

$$R_0(s) \rightarrow_{red} (\lambda x.\lambda y.\lambda z.t_2[\vec{x}'/\vec{r}'])r(\lambda x.s)(\lambda v.R_0((\lambda x.s)v)) \\ \rightarrow_{red} t_2[x/r, y/\lambda x.s, z/(\lambda v.R_0((\lambda x.s)v))]$$

$$R_1(s') \rightarrow_{red} t_2'[x/r', y/\lambda x.s', z/(\lambda v'.R_1((\lambda x'.s')v'))]$$

follows (+), and we are done. (b,ii) follows as in the case $(N_k^{E,=})$.
Case $(+^{E,=})$: Assume $\Gamma^=(\vec{r};\vec{s})$. By IH $t_0[\vec{r}] \rightarrow_{red} i(r) \in Term_{nf}$, $t_0'[\vec{s}] \rightarrow_{red} i(r') \in Term_{nf}$ and $pair(r,r') \in A^*[\vec{r}]$ or $t_0[\vec{r}] \rightarrow_{red} j(r) \in Term_{nf}$, $t_0'[\vec{s}] \rightarrow_{red} j(r') \in Term_{nf}$ and $pair(r,r') \in B^*[\vec{r}]$. Let $[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x\}$. In the first case we have

$$D(t_0, (x)t_1, (y)t_2)[\vec{r}] \rightarrow_{red} (\lambda x.(t_1[\vec{r}']))r \rightarrow_{red} t_1[x/r, \vec{x}/\vec{r}],$$

$$D(t_0', (x)t_1', (y)t_2')[\vec{r}] \rightarrow_{red} t_1'[x/r', \vec{x}/\vec{s}],$$

$$pair(t_1[x/r, \vec{r}], t_1'[x/r', \vec{s}]) \in C[z/i(x)]^*[x/r, \vec{r}] = C^*[z/i(r), \vec{r}]$$

and using arguments as before

$$C^*[z/i(r), \vec{r}] = C^*[z/t_0[\vec{r}], \vec{r}] = C[z/t_0]^*[\vec{r}]$$

and we are done. The second assertion follows in the same way.
(b,ii) follows as before.
Case (I^E): Assume $\Gamma^=(\vec{r};\vec{s})$. By IH 1 follows $(I(A,t_1,t_2))^*[\vec{r}] \neq \emptyset$, $pair(t_1[\vec{r}], t_2[\vec{r}]) \in A^*[\vec{r}]$. Further by IH 3 $pair(t_2[\vec{r}], t_2[\vec{s}]) \in A^*[\vec{r}]$, and by $equiv(A^*[\vec{r}])$ follows (b,i). (b,ii) is trivial.

Case $(\Pi^=)$, $(\Sigma_0^=)$, $(\Sigma_1^=)$: By using the proof for the elimination rules we see, that if the conclusion is $r = s : C$, we conclude assuming $\Gamma^=(\vec{r};\vec{s})$, that $(r = r : C)[\vec{r};\vec{s}]$, further $(r[\vec{s}] \rightarrow_{red} t \in Term_{nf}) \rightarrow (s[\vec{s}] \rightarrow_{red} t)$, therefore follows $(r = s : C)[\vec{r};\vec{s}]$.
Case (Π^η):Assume $\Gamma^=$ By IH we have

$$pair(t[\vec{r}], t[\vec{s}]) \in (\Pi x \in A.B)^*[\vec{r}],$$

therefore $t[\vec{r}]\longrightarrow_{red}\lambda x.s$, $t[\vec{s}]\longrightarrow_{red}\lambda x'.s'$,

$$pair(\lambda x.s, \lambda x'.s') \in (\Pi x \in A.B)^{basis}[\vec{r}],$$

Assume $pair(r, r') \in A^*[\vec{r}]$. Then $r\longrightarrow_{red}\widetilde{r}$, $r'\longrightarrow_{red}\widetilde{r}'$, $pair(\widetilde{r}, \widetilde{r}') \in A^*[\vec{r}]$, $\widetilde{r}, \widetilde{r}' \in Term_{nf}$.

$$Ap(t, x))[\vec{r}][x/r] = Ap(t[\vec{r}], r)\longrightarrow_{red}Ap(\lambda x.s, \widetilde{r})\longrightarrow_{red}s[x/\widetilde{r}],$$

and since

$$pair(s[x/\widetilde{r}], s'[x'/\widetilde{r}']) \in B^*[x/\widetilde{r}, \vec{r}] = B^*[x/r, \vec{r}],$$

follows

$$pair(Ap(t, x)[\vec{r}][x/r], s'[x'/\widetilde{r}']) \in B^*[x/r, \vec{r}],$$

therefore

$$pair(Ap(t, x)[\vec{r}], \lambda x'.s') \in (\Pi x \in A.B)^*[\vec{r}],$$

$$pair(Ap(t, x)[\vec{r}], t[\vec{r}]) \in (\Pi x \in A.B)^*[\vec{r}].$$

Case $(\Sigma_3^=)$: Assume $\Gamma^=(\vec{r}; \vec{s})$. By IH $t[\vec{r}]\longrightarrow_{red}r$, $t[\vec{s}]\longrightarrow_{red}r'$ for some $pair(r, r') \in \Sigma x \in A.B^*[\vec{r}] \cap (Term_{nf} \times Term_{nf})$, $p(p_0(t), p_1(t))[\vec{r}]\longrightarrow_{red}r$ and we are done.

Case $(I^=)$: Assume $\Gamma^=(\vec{r}; \vec{s})$. By IH we conclude $pair(t_0[\vec{r}], t_0[\vec{s}]) \in I(A, t_1, t_2)^*[\vec{r}]$, therefore $t_0[\vec{s}]\longrightarrow_{red}\underline{r}$, $pair(\underline{r}, \underline{r}) \in I(A, t_1, t_2)^*[\vec{r}]$, $pair(t_0[\vec{r}], \underline{r}) \in I(A, t_1, t_2)^*[\vec{r}]$. (b,ii) is trivial.

Case other equality rules: Let $r = s : A$ be the conclusion of the rules. By using several times the rules general rules, elimination rules and in case $W^=$ the introduction rules we can conclude $r = r : A$, and $s = s : A$. (For $(W^=)$ we argue that $\Gamma, v : B[x/t_0] \Rightarrow Ap(t_1, v) : Wx \in A.B$, by $(W^{E,=})$ $\Gamma, v : B[x/t_0] \Rightarrow R(Ap(s', v), (x, y, z)t') : C[u/Ap(s', v)]$, by $\Pi^{I,=}$ $\Gamma \Rightarrow \lambda v.R(Ap(s', v), (x, y, z)t') : \Pi v \in B.C[u/Ap(s', v)]$, by $(ALPHA)$ for the z_i^{ML}, that we need, and it follows $\Gamma \Rightarrow \lambda z_i^{ML}.R(Ap(s', z_i^{ML}), (x, y, z)t') : \Pi v \in B.C[u/Ap(s', v)]$, and now by (SUB) follows the assertion). Now, assuming $\Gamma^=(\vec{r}, \vec{s})$, and using the proofs above we can conclude $pair(r[\vec{r}], r[\vec{s}]) \in A^*[\vec{r}]$ and $A^*[\vec{r}] = A^*[\vec{s}]$, so (b,i). In all the cases, we have, if the right side is written as $t[x_1/r_1, \ldots, x_n/t_n]$, if x_i corresponds to the type B_i (read off from the rule) follows easily by IH and using the proofs of several rules handled before the assertion for $\Gamma \Rightarrow r_i : B_i$, therefore $r_i[\vec{r}]\longrightarrow_{red}\widetilde{r}_i \in Term_{nf}$ for some $\widetilde{r}_i$, $pair(\widetilde{r}_i, r_i[\vec{r}]) \in B_i[\vec{r}]$, further $r[\vec{r}]\longrightarrow_{red}t[x_1/\widetilde{r}_1, \cdots x_n\widetilde{r}_n, \vec{r}]$. We conclude

$$pair(t[x_1/\widetilde{r}_1, \ldots, x_n/\widetilde{r}_n, \vec{r}], t[x_1/r_1[\vec{r}], \ldots, x_n/r_n[\vec{r}], \vec{r}]) \in A^*[\vec{r}].$$

Now using $equiv(A^*[\vec{r}])$ and lemma 6.13 we conclude

$$pair(r[\vec{r}], s[\vec{r}]) \in A^*[\vec{r}], pair(s[\vec{r}], s[\vec{s}]) \in A^*[\vec{r}],$$

and have (b,i).

Case (U^I): trivial.

$(T^{I,=})$ we have by IH, assuming $\Gamma^=(\vec{r}; \vec{s})$,

$$pair(a[\vec{r}], a'[\vec{s}]) \in U^*$$

therefore,

$$tripel(a[\vec{r}], b, a'[\vec{s}]) \in \widehat{U}$$

for some b, by $Cor_U(\widehat{U})$,

$$tripel(a[\vec{r}], b, a[\vec{r}]) \in \widehat{U}, tripel(a[\vec{s}], b, a[\vec{s}]) \in \widehat{U},$$

and

$$T(a)^*[\vec{r}] = b = T(a')^*[\vec{s}]$$

Case ($\underline{n}^I_k$), ($\underline{n}^I$): trivial.
Case ($\pi^{I,=}$): Assume $\Gamma^=(\vec{r};\vec{s})$. By IH $a[\vec{r}]\rightarrow_{red}\widetilde{a}$, $a'[\vec{s}]\rightarrow_{red}\widetilde{a}'$,

$$\exists\gamma < \alpha_I \exists b' \in \ TC(\widetilde{U}^\gamma(\cdot)(\emptyset))(tripel(\widetilde{a}, b', \widetilde{a}), tripel(\widetilde{a}', b', \widetilde{a}) \in \widetilde{U}^\gamma(\cdot)(\emptyset)),$$

and

$$\forall pair(t, t') \in b' \rightarrow \exists\delta < \alpha_I.\exists c \in \ TC(\widetilde{U}^\delta(\cdot)(\emptyset)).$$
$$(tripel(b[x/t, \vec{r}], c, b'[x/t', \vec{s}]) \in \widetilde{U}^\delta(\cdot)(\emptyset)).$$

Since $Ad(L_{\alpha_I})$ (here is the central point where we need (Δ_0-coll) and and admissible a which is closed under the step to the next admissible), and $TC(\widetilde{U}^\beta(\cdot)(\emptyset)) \in L_{\alpha_I}$ ($\beta < \alpha_I$), there is a $\rho < \alpha_I$, such that $\gamma < \rho$ and δ can be chosen $< \rho$. There are now b, f such that ($[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}] \setminus \{x\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{s}] \setminus \{x\}$) $\phi(\widetilde{a}, x, b[\vec{r}'], \widetilde{a}', x, b[\vec{s}'], b, f, Util^\rho(\cdot)(\emptyset))$, (note that the c we used above is correct by $Cor(\widehat{U})$) and by 6.5 follows

$$pair(\pi x \in a.b, u_\pi(b, f), \pi x \in a'.b') \in \widetilde{U}^{\rho+1}(\cdot)(\emptyset).$$

The cases ($\sigma^{I,=}$), ($w^I, =$), ($\widetilde{+}^{I,=}$), ($i^{I,=}$) follow in a similar way.

Cases ($\underline{n}^=_k$), ($\underline{n}^=$) are immediate.
Case ($\pi^=$): Assume $\Gamma^=(\vec{r};\vec{s})$ and chose b', f, ρ as in ($\pi^{I,=}$). Then $T(a)^*[\vec{s}] = b'$, and if $pair(t, t') \in b'$, $T(b)^*[x/t, \vec{r}] = f(t) = f(t') = T(b)^*[x/t', \vec{s}]$. Since we have $Cor(\widehat{U})$ (by lemma 6.9) follows

$$T((\pi x \in a.b))^*[\vec{r}] = u_\pi(b', f) = (\Pi x \in T(a).T(b))^*[\vec{r}].$$

Cases ($\sigma^=$), ($\widetilde{+}^=$) are treated in a similar way.
In the case of ($w^=$) we conclude as before, that $u^\gamma_w(b, f, \cdot)(\emptyset) = F^\gamma(\vec{s}, \cdot)(\emptyset)$, and, since $\alpha(\widetilde{U}^\rho(\cdot)(\emptyset)) < \alpha_I$, ($\rho$ chosen as in ($\pi^{I,=}$)) follows by 6.1
$T(wx \in a.b)^*[\vec{r}] = u_w^{\alpha(\widetilde{U}^\rho(\cdot)(\emptyset))}(b, f, \cdot)(\emptyset) = F^{\alpha_I}(\vec{s}, \cdot)(\emptyset)c = (Wx \in T(a).T(b))^*[\vec{s}]$.

Lemma 6.19 *The Main Lemma 6.18 is valid, if we replace $ML^e_1W_T$ by $ML^e_1W_{T,U}$.*

Proof:
As before, we only hae to check the rules, defined in 3.4 (a):
Case (σ^E): Assume $\Gamma^=(\vec{r};\vec{s})$. By IH exists c, $\alpha \prec \alpha_I$ such that

$$tripel((\sigma x \in a.b)[\vec{r}], c, (\sigma x \in a.b)[\vec{s}]) \in \widetilde{U}^\alpha(\emptyset).$$

Let α be chosen minimal. Then, $\alpha = \alpha' + 1$, and with $u := \widetilde{U}^{\alpha'}$ there exist $r, r' \in Term_{nf}$, c, c', f, f' such that $a[\vec{r}]\rightarrow_{red}r$, $a[\vec{s}]\rightarrow_{red}r'$, $tripel(r, c, r') \in u$ and (with $[\vec{x}'/\vec{r}'] := [\vec{x}/\vec{r}]\setminus\{x\}$, $[\vec{x}'/\vec{s}'] := [\vec{x}/\vec{s}] \setminus \{x\}$) $\forall pair(t, t') \in c.tripel(s[\vec{r}'][x/t], f(t), s[\vec{s}'][x/t']) \in u$. Therefore $T(a[\vec{r}])^* = c = T(a[\vec{s}])^*$, and for $pair(t, t') \in T(a[\vec{r}])^*$, $pair(s[x/t, \vec{r}], s[x/t', \vec{s}]) \in U^*[\,]$.
(π^E), (w^E), ($\widetilde{+}^E$) are checked in the same way. For ($\widetilde{i}^E$) we observe, that $a[\vec{r}]\rightarrow_{red}\widetilde{a}$, $a[\vec{s}]\rightarrow_{red}\widetilde{a}'$, $s[\vec{r}]\rightarrow_{red}\widetilde{s}$, $s[\vec{s}]\rightarrow_{red}\widetilde{s}'$, $t[\vec{r}]\rightarrow_{red}\widetilde{t}$, $t[\vec{s}]\rightarrow_{red}\widetilde{t}'$, and $tripel(\widetilde{a}, c, \widetilde{a}') \in u$, for some u as before, $T(a)^*[\vec{r}] = c$, $pair(\widetilde{s}, \widetilde{s}') \in c$, $pair(\widetilde{t}, \widetilde{t}') \in c$, and since c is closed under $\rightarrow_{red}$ follows the assertion.

Chapter 7

Arithmetical formulas in $ML_1^eW_T$ and KPi^+

In this chapter we want to evaluate the results we have found out to get the proof theoretical strength of Martin-Löf's type theory. We will interpret the language of Peano Arithmetic (L_{PA}, introduced in 7.1) in L_{ML} and L_{KP} (definition7.2) and prove that it permutes with the interpretation of Martin-Löf's type theory in KPi^+ (lemmata 7.4 and 7.6). Next we observe, that we could interpret every proof in some theory KPi_n^+, and have a stronger bound (lemma 7.7). At the end we analyze the proof theoretical strength of KPi_n^+ and have the desired upper bound (theorem 7.8).

Definition 7.1 *Definition of the language of Peano Arithmetic L_{PA}: Variables should be a set $Var_{PA} := \{v_i^{PA} | i \in I\!N\}$, $v_i^{PA} \neq v_j^{PA}$ for $i \neq j$. Further we have symbols for each primitive recursive function, $=$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\bot$, and $.$, $,$, $($, $)$.*
Terms are Variables and $f(t_1, \ldots, t_n)$ if t_i are terms and f is a symbol for a n-ary primitive recursive function.
Prime formulas are $\bot$ and equations $r = s$ for r, s terms.
Formulas are prime formulas and $A \rightarrow B$, $A \wedge B$, $A \vee B$, $\forall x.A$, $\exists x.A$, if A, B formulas, $x \in Var_{PA}$.

Definition 7.2 *(a) For each primitive recursive $g : I\!N^k \rightarrow I\!N$ we define a closed g-term $int_{PA,ML}(g)$, (we abbreviate this as $\hat{g} := int_{PA,ML}(g)$) such that*

$$ML_1^eW_T \vdash \hat{g} : \underbrace{N \rightarrow \cdots N}_{k \ times} \rightarrow N,$$

and we define a set $int_{PA,KP}(g)$ short $\tilde{g}$ in L_{KP}
such that $KPi^+ \vdash fun(\tilde{g}) \wedge dom(\tilde{g}) = I\!N^k \wedge \forall x \in I\!N^k.\tilde{g}(x) \in I\!N$.
Case $g = S$: $\hat{g} := \lambda x.Sx$, $\tilde{g} := \{pair(x, x+1) | x \in I\!N\}$.
Case $g = Proj_i^n$:
$\hat{g} := \lambda x_1, \ldots, x_n.x_i$, $\tilde{g} := \{pair(tupel(x_1, \ldots, x_n), x_i) | x_1, \ldots, x_n \in I\!N\}$.
Case $g = Cons_c^n$:
$\hat{g} := \lambda x_1, \ldots, x_n.S^c0$, $\tilde{g} := \{pair(tupel(x_1, \ldots, x_n), c) | x_1, \ldots, x_n \in I\!N\}$.
Case $g(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$:
$\hat{g} := \lambda x_1, \ldots, x_n.\hat{h}(\hat{g}_1 x_1 \cdots x_n) \cdots (\hat{g}_m x_1 \cdots x_n)$,
$\tilde{g} := \{pair(tupel(x_1, \ldots, x_n), h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)))$
$| x_1, \ldots, x_n \in I\!N\}$.
Case $g(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$,
$g(x_1, \ldots, x_n, y+1) = k(x_1, \ldots, x_n, y, g(x_1, \ldots, x_n, y))$:

$\hat{g} := \lambda x_1, \ldots, x_n, y.P(y, \hat{h}x_1 \cdots x_n, (u, v)\hat{k}x_1 \cdots x_n uv)$,
define $a(x_1, \ldots, x_n, 0) := \tilde{h}(x_1, \ldots, x_n)$,
$a(x_1, \ldots, x_n, Sy) := \tilde{k}(x_1, \ldots, x_n, y, a(x_1 \cdots x_n, y))$,
then
$\tilde{g} := \{pair(tupel(x_1, \ldots, x_n, y), a(x_1, \ldots, x_n, y)) | x_1, \ldots, x_n, y \in I\!N\}$.

(b) *For each term* t *of PA we define a g-term* $int_{PA,ML}(t)$, *short* $\hat{t}$ *and a term of* L_{KP} $int_{PA,KP}(t)$, *short* $\tilde{t}$, *such that, if* $FV(t) = \{v_{i_1}^{PA}, \ldots, v_{i_n}^{PA}\}$ $(i_1 < \cdots i_n)$ (v_i^{PA} *as in definition 7.1 of* Var_{PA}) *then* $FV(\hat{t}) \subset \{z_{i_1}^{ML}, \ldots, z_{i_n}^{ML}\}$, $FV(\tilde{t}) \subset \{u_{i_1}^{KP}, \ldots, u_{i_n}^{KP}\}$ *and* $ML_1^e W_T \vdash z_{i_1}^{ML} : N, \ldots, z_{i_n}^{ML} : N \Rightarrow \hat{t} : N$, *and*
$KPi^+ \vdash \forall u_{i_1}^{KP}, \ldots, u_{i_n}^{KP} \in I\!N.(\tilde{t} \in I\!N)$.
Case $t = v_i^{PA}$: $\hat{t} := z_i^{ML}$, $\tilde{t} := u_i^{KP}$.
Case $t = 0$: $\hat{t} := 0$, $\tilde{t} := 0$.
Case $t = gt_1 \cdots t_n$: $\hat{t} := \hat{g}\hat{t}_1 \cdots \hat{t}_n$, $\tilde{t} := \tilde{g}(\tilde{t}_1, \ldots, \tilde{t}_n)$.

(c) *For each formula* A *of PA we define a g-type* $int_{PA,ML}(A)$, *short* $\widehat{A}$, *and a formula of* L_{KP} $int_{PA,KP}(A)$, *short* $\widetilde{A}$, *such that in* KPi^+ $\widetilde{A}$ *is equivalent to a* Δ_0*-formula, and if* $FV(A) = \{v_{i_1}^{PA}, \ldots, v_{i_n}^{PA}\}$ $(i_1 < \cdots i_n)$
then $FV(\widehat{A}) \subset \{z_{i_1}^{ML}, \ldots, z_{i_n}^{ML}\}$, $FV(\widetilde{A}) \subset \{u_{i_1}^{KP}, \ldots, u_{i_n}^{KP}\}$ *and*
$ML_1^e W_T \vdash z_{i_1}^{ML} : N, \ldots, z_{i_n}^{ML} : N \Rightarrow \widehat{A}$ *type*.
Case $A = (s = t)$: $\widehat{A} := I(N, \hat{s}, \hat{t})$, $\widetilde{A} := (\tilde{s} = \tilde{t})$.
Case $A = (B \wedge C)$: $\widehat{A} := (\widehat{B} \times \widehat{C})$, $\widetilde{A} := \widetilde{B} \wedge \widetilde{C}$.
Case $A = (B \vee C)$: $\widehat{A} := (\widehat{B} + \widehat{C})$, $\widetilde{A} := (\widetilde{B} \vee \widetilde{C})$.
Case $A = (B \to C)$: $\widehat{A} := (\widehat{B} \to \widehat{C})$, $\widetilde{A} := (\widetilde{B} \to \widetilde{C})$.
Case $A = \forall v_i^{PA}.B$: $\widehat{A} := \Pi z_i^{ML} \in N.\widehat{B}$, $\widetilde{A} := \forall u_i^{KP} \in I\!N.\widetilde{B}$.
Case $A = \exists v_i^{PA}.B$: $\widehat{A} := \Sigma z_i^{ML} \in N.\widehat{B}$, $\widetilde{A} := \exists u_i^{KP} \in I\!N.\widetilde{B}$.
Case $A = \bot$: $\widehat{A} := N_0$, $\widetilde{A} := (0 \neq 0)$.

Definition 7.3 (a) *We define* $emb : I\!N \to I\!N$, $embnat(n) := S^n 0 (=: \hat{n})$ *(or more precisely* $\ulcorner S^n 0 \urcorner$*), a function definable in* KPi^+.

(b) $makepair(a) := pair(a, a)$.

Lemma 7.4 (a) *If* $g : I\!N^k \to I\!N$ *is primitive recursive, then*

$$KPi^+ \vdash \quad \forall t_1, \ldots, t_k \in Term_{Cl}.\forall n_1, \ldots, n_k. \\ (r^1 \to_{red} \hat{n}_1 \wedge \cdots \wedge r^k \to_{red} \hat{n}_k) \to \hat{g}r_1, \ldots, r_k \to_{red} emb(\tilde{g}(n_1, \ldots, n_k)).$$

(b) *If* t *is a term of PA,* $FV(t) \subset \{v_1^{PA}, \ldots, v_n^{PA}\}$, *then*

$$KPi^+ \vdash \quad \forall r_1, \ldots, r_n \in Term_{Cl}.\forall n_1, \ldots, n_k.(r^1 \to_{red} \hat{n}_1 \wedge \cdots \wedge r^k \to_{red} \hat{n}_k) \to \\ \hat{t}[z_1^{ML}/r_1, \ldots, z_n^{ML}/r_n] \to_{red} emb(\tilde{t}[u_1^{KP}/n_1, \ldots, u_n^{KP}/n_n]).$$

Proof: (a) Case $g = S$: $\hat{g}t_1 \to_{red} (\lambda x.Sx)\hat{n}_1 \to_{red} S\hat{n}_1 = emb(Sn_1)$.
Case $g = Proj_i n$: $\hat{g}t_1, \ldots, t_j \to_{red} \lambda x_{i+1}, \ldots, x_n.x_i$ for $j < i$, $\lambda x_{i+1}, \ldots, x_n.\hat{n}_i$ for $i \leq j$.
Case $g = Cons_c^n$: trivial.
Case $g(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$, $l_i := \tilde{g}_i(n_1, \ldots, n_n)$:

$$\hat{g}_i(\hat{n}_1, \ldots, \hat{n}_n) \to_{red} emb(\tilde{g}_i(n_1, \ldots, n_n)) = emb(l_i),$$

therefore

$$\begin{array}{rl} & gr_1,\ldots,r_n \\ \to_{red} & hg_1(\hat{n}_1,\ldots,\hat{n}_n),\ldots,g_m(\hat{n}_1,\ldots,\hat{n}_n) \\ \to_{red} & emb(\widetilde{h}(l_1,\ldots,l_m)) \\ = & embnat(g(n_1,\ldots,n_n)) \end{array}$$

Case $g(x_1,\ldots,x_n,0) = h(x_1,\ldots,x_n)$, $g(\vec{x},x_{n+1}) = k(\vec{x},y,g(\vec{x},x_{n+1})))$. Induction on n_{n+1}. If $n_{n+1} = 0$,

$$\begin{array}{rl} & g(r_1,\ldots,r_n,r_{n+1}) \\ \to_{red} & P(n_{n+1},(\hat{h}\hat{n}_1,\ldots,\hat{n}_n),(\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)) \\ \to_{red} & P(n_{n+1},emb(\widetilde{h}(n_1,\ldots,n_n)),(\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)) \\ \to_{red} & emb(\widetilde{h}(n_1,\ldots,n_n)) \end{array}$$

If $n_{n+1} = k+1$,

$$\begin{array}{rl} & g(t_1,\ldots,t_n,t_{n+1}) \\ \to_{red} & P(S\hat{k},(\hat{h}\hat{n}_1,\ldots,\hat{n}_n),(\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)) \\ \to_{red} & P(S\hat{k},emb(\widetilde{h}(n_1,\ldots,n_n)),(\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)) \\ \to_{red} & (\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)\hat{k}P(\hat{k},emb(\widetilde{h}(n_1,\ldots,n_n)),(\lambda u,v.\hat{k}\hat{n}_1,\ldots,\hat{n}_n uv)) \\ \to_{red} & \hat{k}\hat{n}_1,\ldots,\hat{n}_n\hat{k}embnat(\widetilde{g}(n_1,\ldots,n_n,k)) \\ \to_{red} & emb(\widetilde{k}(n_1,\ldots,n_n,k,\widetilde{g}(n_1,\ldots,n_n,k))) = emb(g(n_1,\ldots,n_n,n_{n+1})) \end{array}$$

(b): If $t = v_i^{PA}, 0$, this is trivial,
and if $t = gt_1,\ldots,t_n$ follows

$$t_i[\vec{r}]\to_{red}emb(\widetilde{t}_i[\vec{n}]),$$

by (a) therefore

$$t[\vec{r}]\to_{red}\widetilde{g}(\widetilde{t}_1[\vec{r}],\ldots,\widetilde{t}_n[\vec{r}]) = \widetilde{t}[\vec{r}]$$

Next task would now be to prove, that, when we first interpret a formula of L_{PA} in L_{ML} and then use the interpretation, as we have done in chapter 5, we get an equivalent formula to the one, we get by directly interpreting L_{PA} in L_{KP}. But in this formulation, this is not correct, here is the place, where we need to extend the set of term constructors by the constructors A_i: Assume, that we had only interpreted $\Pi x \in A.B$ as a set of closed terms, which all represent recursive functions. Let $A(x,y)$ be a prime formula of L_{PA}, such that, $KPi^+ \vdash int_{PA,KP}(\forall x.\exists y.A(x,y))$ and such that there is no recursive function g such that $\forall x.A(x,g(x))$. Then we had $int_{PA,ML}(\forall x.\exists y.A(x,y))^*[\ ] = \emptyset$, so for every $t \in Term$, $FV(t) \subset \{x\}$ $pair(\lambda x.t,\lambda x.t) \in int_{PA,ML}((\forall x.\exists y.A(x,y)) \to\perp)^*[\ ]$, although $KPi^+ \vdash \neg int_{PA,KP}((\forall x.\exists y.A(x,y)) \to\perp)$. But when we add a constructor A_i together with its interpretation A_i^* such that $\forall x.(\exists y.A(x,y)) \to A(x,A_i^*(x))$, we have $int_{PA,ML}(\forall x.\exists y.A(x,y))^*[\ ] \neq \emptyset$ and $int_{PA,ML}(\forall x.\exists y.A(x,y) \to\perp)^*[\ ] = \emptyset$. Following this idea we can define for every formula a finite sequence of functions $f_1,\ldots,f_n$, such that the two interpretations are equivalent, if we have A_i with $A_i^* = f_i$, as will be stated in the next lemma:

Lemma 7.5 *For every formula A of PA with*

$$FV(A) \subset \{v_{i_1}^{PA},\ldots,v_{i_l}^{PA}\}$$

with $i_1 < \cdots i_l$ there exist a finite set of functions F, with $\forall f \in F.\exists n.f : I\!N^n \to I\!N$, definable in KPi^+, such that for every interpretation relative to an extending set of constructors $(A_i)_{i\in I}$ with interpretations $(A_i^*)_{i\in I}$, such that $F \subset \{A_i^* | i \in I\}$ there exists a $h \in Term_{(A_i,A_i^*)_{i\in I}}$ $FV(t) \subset \{z_{i_1}^{ML}, \ldots, z_{i_l}^{ML}\}$, ($\vec{z} := z_{i_1}^{ML}, \ldots, z_{i_l}^{ML}$, $\vec{u} := u_{i_1}^{KP}, \ldots, u_{i_l}^{KP}$) such that

$$\begin{array}{ll} KPi_n^+ \vdash & \forall n_1, \ldots, n_l \in I\!N.\forall r_1, \ldots, r_l \in Term_{Cl}. \\ & \quad (r_1 \to_{red} \hat{n}_1 \wedge \cdots \wedge r_l \to_{red} \hat{n}_l) \to \\ & \quad\quad ((\exists r \in Term_{nf}.h[\vec{z}/\vec{r}] \to_{red} r) \wedge \\ & \quad\quad (\widetilde{A}[\vec{u}/\vec{n}] \leftrightarrow makepair(h[\vec{z}/\vec{r}]) \in \widehat{A}^*[\vec{z}/\vec{r}]) \wedge \\ & \quad\quad (\widetilde{A}[\vec{u}/\vec{n}] \leftrightarrow \widehat{A}^*[\vec{z}/\vec{r}] \neq \emptyset)) \end{array}$$

Proof: by induction on the definition of the formulas.
Again we will not mention explicitly Variables, that occur in subterms, or do not occur at all. Case $A = \perp$: Choose $F := \emptyset$, $h := 0$ We have $\neg\widetilde{A}[\vec{n}]$, $\widehat{A}^*[\vec{r}] = \emptyset$.

Case $A = (s = t)$: Choose as $F := \emptyset$, $h := \underline{r} \in Term_{nf}$. We have, using that for $r \to_{red} s \in Term_{nf}$ s is unique, and 7.4

$$\begin{array}{lll} \widetilde{A}[\vec{n}] & \leftrightarrow & \widetilde{s}[\vec{n}] = \widetilde{t}[\vec{n}] \\ & \leftrightarrow & (\exists n \in \mathbb{N}.\hat{s}[\vec{r}] \to_{red} S^n 0 \wedge \hat{t}[\vec{r}] \to_{red} S^n 0) \\ & \leftrightarrow & pair(\hat{s}[\vec{r}], \hat{t}[\vec{r}]) \in N^* \\ & \leftrightarrow & pair(\underline{r}, \underline{r}) \in \widehat{A}^*[\vec{r}] \\ & \leftrightarrow & \widehat{A}^*[\vec{r}] \neq \emptyset \end{array}$$

Case $A = (B \wedge C)$: Let F_i for A_i chosen, $F := F_1 \cup F_2$. If $F \subset \{A_i^* | i \in I\}$, and we have h_i according to the assertion of the lemma chosen. Then we define $h := p(h_1, h_2)$. Then for $\vec{n} \in \mathbb{N}$, $\vec{r} \in Term_{Cl}$ as in the lemma, exist $s_1, s_2 \in Term_{nf}$, such that $r_i[\vec{r}] \to_{red} s_i$, $h[\vec{r}] \to_{red} p(s_1, s_2) \in Term_{nf}$.

$$\begin{array}{lll} \widetilde{A}[\vec{n}] & \leftrightarrow & \widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}] \\ & \leftrightarrow & makepair(s_1) \in \widehat{B}_1^*[\vec{n}] \wedge makepair(s_2) \in \widehat{B}_2^*[\vec{n}] \\ & \leftrightarrow & makepair(h[\vec{r}]) \in \widehat{A}^*[\vec{n}] \\ & \leftrightarrow & \widehat{B}_1^*[\vec{r}] \neq \emptyset \wedge \widehat{B}_2^*[\vec{r}] \neq \emptyset \\ & \leftrightarrow & \widehat{A}^*[\vec{n}] \neq \emptyset \end{array}$$

Case $A = (B_1 \vee B_2)$: Let F_i for B_i chosen. Let

$$f_3 := \{pair(tupel(n_1, \ldots, n_l), i) | (i = 0 \wedge \widetilde{B}_0[\vec{n}]) \vee (i = 1 \wedge \neg\widetilde{B}_0[\vec{n}])\}$$

$F := F_1 \cup F_2 \cup \{f_3\}$, and if $F \subset \{A_i^* | i \in I\}$, further $A_k^* = f_3$, h_i chosen according to the IH for F and define $h := P(A_k(\vec{z}), h_1, (u, v)h_2)$ (u, v new variables). Then if $\vec{n}$, $\vec{r}$ are chosen as in the assertion, there exist s_i such that $h_i[\vec{r}] \to_{red} s_i \in Term_{nf}$

$$A_k(r_1, \ldots, r_n) \to_{red} A_k(\hat{n}_1, \ldots, \hat{n}_i) \to_{red} S^i 0$$

for $i = f_3(\vec{n}) \in \{0, 1\}$, if $i = 0$,

$$h[\vec{r}] \to_{red} P(S^i 0, s_1, (u, v)h_2[\vec{r}]) \to_{red} s_1 \in Term_{nf},$$

and if $i = 1$,

$$h[\vec{r}] \rightarrow_{red} s_2.$$

We have

$$\begin{array}{rcl} \widetilde{A}[\vec{n}] & \leftrightarrow & \widetilde{B}_1[\vec{n}] \vee (\neg \widetilde{B}_1[\vec{n}] \wedge \widetilde{B}_2[\vec{n}]) \\ & \leftrightarrow & (f_3(\vec{n}) = 0 \wedge makepair(s_1) \in \widehat{B}_1^*[\vec{r}]) \vee (f_3(\vec{n}) = 1 \wedge makepair(s_2) \in \widehat{B}_2^*[\vec{r}]) \\ & \leftrightarrow & makepair(h[\vec{r}]) \in \widehat{A}^*[\vec{r}] \\ & \leftrightarrow & \widehat{B}_1^*[\vec{r}] \neq \emptyset \vee \widehat{B}_2^*[\vec{r}] \neq \emptyset \\ & \leftrightarrow & \widehat{A}^*[\vec{r}] \neq \emptyset \end{array}$$

Case $A = (B_1 \rightarrow B_2)$. Let F_i for B_i chosen, $F := F_1 \cup F_2$, $F \subset \{A_i^* | i \in I\}$, and assume that h_i are chosen for A_i. Define $h := \lambda x.h_2$ Then for $\vec{r}$, $\vec{n}$ as in the assertion $h[\vec{r}] \in Term_{nf}$, By IH $h_2[\vec{r}] \rightarrow_{red} s_2$ for some $s_2 \in Term_{nf}$.
Subcase $\widetilde{A}[\vec{n}]$. If $\widetilde{B}_1[\vec{n}]$ is false, then by IH $\widehat{B}_1^*[\vec{r}] = \emptyset$, therefore

$$\forall pair(r, r') \in \widehat{B}_1^*[\vec{r}].pair(h_2[x/r, \vec{r}], h_2[x/r', \vec{r}]) \in \widehat{B}_2^*[\vec{r}]$$

therefore $makepair(h[\vec{r}]) \in \widehat{A}^*[\vec{r}]$.
If $\widetilde{B}_1[\vec{n}]$ is true, then $\widetilde{B}_2[\vec{r}]$ is true, therefore $makepair(s_2) \in \widehat{B}_2^*[\vec{r}]$,

$$\forall pair(r, r') \in \widehat{B}_1^*[\vec{r}].h_2[x/r, \vec{r}] \rightarrow_{red} s_2 \wedge h_2[x/r', \vec{r}] \rightarrow_{red} s_2 \wedge makepair(s_2) \in \widehat{B}_2^*[\vec{n}],$$

$h[\vec{r}] \in \widehat{A}^*[\vec{n}]$.
Subcase $\neg\widetilde{A}[\vec{n}]$. Then by IH exists s_1 such that $h_1[\vec{r}] \rightarrow_{red} s_1 \in Term_{nf}$ and we have $makepair(s_1) in \widehat{B}_1^*[\vec{r}]$ and, if we had $pair(s, s') \in \widehat{A}^*[\vec{r}]$, then $makepair(s) \in \widehat{A}^*[\vec{r}]$, $s \rightarrow_{red} \lambda x.t$ for some t, $makepair(t[x/s_1]) \in \widehat{B}_2^*[\vec{r}] = \emptyset$, a contradiction, therefore $\widehat{A}^*[\vec{r}] = \emptyset$.
Case $A = \forall v_i^{PA}.B$: Let F_1 for B be chosen, $F := F_1$, $F \subset \{A_i^* | i \in I\}$, h_1 be chosen for B, and let $h := \lambda v_i^{PA}.h_1$. Assume $\vec{n}$, $\vec{r}$ as in the assertion, $h[\vec{r}] \in Term_{nf}$.
Assume $pair(r, r') \in \widehat{A}^*[\vec{r}]$, then $pair(r, r) \in \widehat{A}^*[\vec{r}]$, $r \rightarrow_{red} \lambda x.t$ and

$$\forall k \in \mathbb{N}.makepair(r[x/\hat{k}, \vec{r}]). \in \widehat{B}^*[z_i^{ML}/\hat{k}, \vec{r}],$$

by IH follows $\forall k \in \mathbb{N}.\widetilde{B}[u_i^{KP}/k, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then for all $k \in \mathbb{N}$ $\widetilde{B}[v_i^{PA}/k, \vec{r}]$, therefore by I.H. $makepair(h[v_i^{PA}/r, \vec{r}]) \in \widehat{B}^*[z_i^{ML}/r, \vec{r}]$, whenever $r \rightarrow_{red} S^k 0$, therefore $makepair(h[\vec{r}]) \in \widehat{A}[\vec{r}]$.
Case $A = \exists v_i^{PA}.B$, F_1 be chosen for B,

$$\begin{array}{rl} f_2 := \{pair(tupel(\vec{n}), k) | & (\widetilde{B}[u_i^{KP}/k, \vec{n}] \wedge \forall k' < k.\neg(\widetilde{B}[u_i^{KP}/k, \vec{n}])) \vee \\ & (k = 0 \wedge \forall k \in \mathbb{N}.\neg(\widetilde{B}[u_i^{KP}/k, \vec{n}]))\} \end{array}$$

Let $F := F_1 \cup \{f_2\}$. Let $F \subset \{A_i^* | i \in I\}$, h_1 be chosen for B and $A_k^* = f_2$. $h := p(A_k(\vec{z}), h_1[z_i^{ML}/A_k(\vec{z})])$ Assume $\vec{n}$, $\vec{r}$ as in the assertion, $k := f_2(\vec{n})$.

$$A_i(\vec{r}) \rightarrow_{red} A_i(\hat{n}_1, \ldots, \hat{n}_n) \rightarrow_{red} S^k 0.$$

By IH we have $h_1[z_i^{ML}/A_k(\vec{z})][\vec{r}] = h_1[z_i^{ML}/A_i(\vec{r}), \vec{r}] \rightarrow_{red} t_1$ for for some $t_1 \in Term_{nf}$, therefore $h[\vec{r}] \rightarrow_{red} p(S^k 0, t_1)$.
Assume $pair(r, r') \in \widehat{A}^*[\vec{r}]$. Then $pair(r, r) \in \widehat{A}^*[\vec{r}]$, $r \rightarrow_{red} p(S^l 0, r'') \in Term_{nf}$. Then $makepair(r'') \in \widehat{B}^*[z_i^{ML}/S^l 0, \vec{r}]$, by I.H. $\widetilde{B}[u_i^{KP}/l, \vec{n}]$, therefore $\widetilde{A}[\vec{n}]$.
Assume $\widetilde{A}[\vec{n}]$. Then by definition $\widetilde{B}[u_i^{KP}/k, \vec{n}]$ and by IH

$$makepair(t_1) \in \widehat{B}^*[z_i^{ML}/\hat{k}, \vec{r}] = \widehat{B}^*[z_i^{ML}/A_k(\vec{r}), \vec{r}] = \widehat{B}[z_i^{ML}/A_k(\vec{z})]^*[\vec{z}/\vec{r}],$$

(we use that $ML_1^e W_T \vdash \vec{z} : N \Rightarrow A$ type,
$pair(A_k(\vec{r}), \hat{k}) \in N^*$ and the Main Lemma), therefore $makepair(h[\vec{r}]) \in \widehat{A}^*[\vec{r}]$.

Lemma 7.6 *If A is a formula in PA, $ML_1^eW_{T,U} \vdash s : \widehat{A}$, then $KPi^+ \vdash \widetilde{A}$.*
Proof: lemma 7.5

Now we observe, that, for a particular proof, the interpretation of $ML_1^eW_{T,U}$ can be carried out in KPi_n^+ for some n: In definition 5.8, we can define A^* in KPi_n^+ for $n > level(A)$, where $level(A)$ counts the nesting of W-types (we replace $a(u)$ by $a(u)^n$, $\alpha(u)$ by $\alpha(u)^n$). (In fact, we only had to count nestings of W-types above U, but this is technically more complicated.) $\widehat{U}$ can be defined, by replacing again $a(u)$, $\alpha(u)$ by $a(u)^n$, $\alpha(u)^n$, in KPi_n^+ for $n \geq 1$. The small lemmata of chapter 6 can be proven in KPi_n^+ for $n \geq 1$ and in lemma 6.18, we can for each judgement prove the conclusion in KPi_n^+ for $n > n_0$, where n_0 is the maximum of the levels of types occurring in the proof in $ML_1^eW_T$. All arithmetical formulas have level 0. So we have the following stronger lemma:

Lemma 7.7 *If A is a formula of PA $ML_1^eW_{T,U} \vdash s : \widehat{A}$, then there exists $n < \mathbb{N}$, such that $KPi_n^+ \vdash \widetilde{A}$.*

Theorem 7.8 $|ML_1^eW_{T,U}|$, $|ML_1^eW_T|$, $|ML_1^eW_{R,U}|$, $|ML_1^eW_R|$, $|ML_1^iW_T|$, $|ML_1^iW_R| \leq \psi_{\Omega_1}(\Omega_{I+\omega})$, *where the ordinal denotation is as in [Buc92b].*

Proof: We show, that $|KPi_n^+| \leq \psi_{\Omega_1}(\epsilon_{\Omega_{I+n}+1}) \leq \psi_{\Omega_1}(\Omega_{I+\omega})$. By lemma 7.6 follows the assertion.
We follow the lines of [Buc92b]. First observe, that we can prove as in theorem 2.9 there, using several applications of $\exists_{\kappa''}$, $(\bigwedge)^*$, and $\vdash^* Ad(L_{\kappa''})$ for $\kappa'' \in \{\kappa, \ldots, \kappa_{n1}\}$, that if:

$$(*) \qquad \lambda \in Lim \quad \wedge \quad \exists \kappa_1, \ldots, \kappa_{n-1}, \kappa \in R.(\forall \alpha < \kappa . \exists \kappa' \in R. \alpha < \kappa' < \kappa) \wedge$$
$$(*) \qquad \kappa \in \kappa_1 \wedge \kappa_1 \in \kappa_2 \wedge \cdots \wedge \kappa_{n-2} \in \kappa_{n-1} \wedge \kappa_{n-1} \in \lambda.$$

and if we extend X^* by $\kappa, \kappa_1, \ldots, \kappa_{n-1}$
follows

$$\vdash^*_\lambda (KPi_n^+)^\lambda.$$

We can adjust theorem 3.12 of [Buc92b] to obtain, if we have $(*)$, $\lambda \in R$, and λ, κ, κ_1, …, $\kappa_{n-1} \in \mathcal{H}$, and $\mathcal{H}$ closed under $\xi \mapsto \xi^R$, then:

For each theorem ϕ of KPi_n^+ exist $k \in \mathbb{N}$ such that $\mathcal{H} \vdash^{\omega^{\lambda+k}}_{\lambda+k} \phi^\lambda$.

Now observe, that $\mathcal{H}_\gamma$ in [Buc92b] has the desired properties (with $\lambda := \Omega_{I+n}$, $\kappa_i := \Omega_{I+i}$) and we conclude as in theorem 4.9, Corollary, with $\nu := \psi_{\Omega_1}(\epsilon_{\Omega_{I+n}})$

$$|KPi_n^+| \leq \psi_{\Omega_1}(\epsilon_{\Omega_{I+n}})$$

Part III

A lower bound for the proof theoretical strength of Martin-Löf's type theory

Chapter 8

A well-ordering proof in Martin-Löf's type theory

In this chapter we will prove

$$ML_1^i W_R \vdash r : \forall \phi \in N \to U.(\quad \forall y \quad \in N.(\forall x \prec y.\phi x) \to \phi y)$$
$$\to \quad \forall y \prec D_{\Omega_{1_{OT}}} \Omega_{I+n}.\phi y$$

for some term r, that is $ML_1^i W_R$ proves the well-ordering up to every ordinal less than our desired proof theoretical strength of $ML_1^i W_R$, $ML_1^i W_T$ and $ML_1^e W_R$. To prove this, we need in some sense to write a computer program, the term r. To do this we will introduce some useful abbreviations.

We will define in Martin-Löf's type theory the analogue of many constructions known from analysis, such that we can later on more or less follow the lines of a well-ordering proof in $\Delta_2^1 - CA + BI$.

After giving some general abbreviations (8.1), we define the type of truth values $\mathcal{B}$ (8.2), the natural numbers (8.3), lists (8.4) and state the obvious properties (lemma 8.5). On page 67 we introduce the four kinds of candidates for the power set of the natural numbers we have, and introduce them in definition 8.6. Then we introduce the subtree ordering (definition 8.8, properties are shown in 8.9), which, although looking very simple, was one of the key ideas in this proof, since it allowed to define $W(X)$ (definition 8.11). On page 73, we explain the general method for the well-ordering proof. We assume some properties for the ordinals (general assumption 8.10) define now $W(X)$ (definition 8.11). We prove, that we have induction over $W(X)$ (8.12), and show some easy properties of $W(X)$ (8.13 and 8.14.

Next we define the "ausgezeichnete Mengen" (definition 8.16). We show, that ausgezeichnete Mengen are segments of each other in the sense of lemma 8.19, define $\mathcal{W}$ (definition 8.20), which is the union of all ausgezeichnete Mengen, and the segments of which are exactly the ausgezeichnete Mengen (lemma 8.21). We show that $\mathcal{W}$ is closed under + and the step to the next cardinal (8.24), contains the type of every element (8.25) and that $\mathcal{W}$ is a "ausgezeichnete Klasse" (definition 8.27). To prove that $\mathcal{W}$ is closed under certain ordinal functions, we need the fundamental sequences of it, stated in 8.10. We prove that, if $I\ \eta\ X$ and $Ag(X)$, then $W(X)$ is closed under $\lambda z.\Omega_z$ ($z \prec I$), (8.30), that $W(X)$ is closed under Veblen-function (8.29) and collapsing function (8.31). Now we define classes, which are syntactical increasing ausgezeichnete Klassen (definition 8.32) and show that we can prove using $\mathcal{W}_{n+1}$ transfinite induction up to $D_{\Omega_1}(\Omega_{I+n}$ (theorem 8.34).

Preliminaries 8.1 *In this part we use the formulation à là Russell for the Universe, where the elements of the universe are types, rather than terms representing indices for types, as in the formulation à là Tarski. In the first part, we preferred the formulation à là Tarski, since indices for the elements of the Universe are better for the interpration in* KPi^+*(we could interpret the Universe as a subset of the natural numbers), whereas the Russel-formulation is more suitable for actual using it (we do not have to apply the type constructor* T *to make a type out of an element of the Universe). The author thinks that it would have been able to carry out the well ordering proof in* $ML_1^iW_T$ *rather than* $ML_1^iW_R$ *— this way he would have proven the proof theoretical equivalence of four, and not only three theories. (we have not proven a lower bound for* $ML_1^iW_T$*) Unfortunately, there was at the end no time to check this any more.*

We use the following abbreviations:
$r0 := p_0(r)$, $r1 := p_1(r)$,
$< r, s >:= p(r, s)$,
$rs := Ap(r, s)$,
$(r =_A s) := I(A, r, s)$
$\forall x \in A.B := \Pi x \in A.B$, $\exists x \in A.B := \Sigma x \in A.B$, *we will use this, if the intended meaning of the type is a formula.*
$\mathcal{B} := N_2$, $\bot := N_0$, $\top := N_1$.
$A \vee B := A + B$, $A \leftrightarrow B := (A \to B) \wedge (B \to A)$, $\neg A := A \to \bot$, $(r \neq_A s) := \neg(r =_A s)$.
$\wedge$, $\vee$, $\forall$, $\exists$ *are used for talking of types, which have formulas as the inteded meaning, whereas the use of* $\times$, $+$, Π, Σ, *indicates, that we are talking of functions and sets.*
We will write $\lambda x, y.t$ *for* $\lambda x.\lambda y.t$, $\forall x, y \in A.B$ *for* $\forall x \in A.\forall y \in A.B$, *similarly for* $\exists, \Pi, \Sigma, W$ *and for more than two variables.*

We will in the following argue informally, especially, if we say: "we have A", or shorter "A", for some g-type A, we mean, there exists a term r such that $ML_1^iW_R \vdash\Rightarrow r : A$. *We write* $r : A$ *for* $ML_1^iW_R \vdash r : A$ *and* $r, s : A$ *for* $ML_1^iW_R \vdash r : A$, *further* $ML_1^iW_R \vdash s : A$, $\Gamma \Rightarrow r : A$ *for* $ML_1^iW_R \vdash \Gamma \Rightarrow r : A$, *etc.*

We will not be very restrictive in the choice of variables, so we will use x, y, z, a, b, c *and sometimes capital letters such that* A, B, C *and* X, Y, Z *for them. We will prefer* i, j, k, n, m *to indicate natural numbers (considered as natural numbers and not as elements of the subsets of* N T'', T', OT, A'' *etc. which are denotations for ordinals),* α, β, γ *for trees (elements of a* W*-type) but sometimes as well for elements of* OT. *Elements of* OT *are usually denoted by* a, b, c *or* x, y, z.

Definition 8.2 *We define some functions corresponding to the type* $\mathcal{B}$, *the type of truth values.*
For $t : \mathcal{B}$ *we define* $\underline{if}\ t\ \underline{then}\ A\ \underline{else}\ B := C_2(t, A, B)$.
$tt := 0_2$, $ff := 1_2$.
$\wedge_{\mathcal{B}} := \lambda x, y.\ \underline{if}\ x\ \underline{then}\ (\underline{if}\ y\ \underline{then}\ tt\ \underline{else}\ ff)\ \underline{else}\ ff$,
$\vee_{\mathcal{B}} := \lambda x, y.\ \underline{if}\ x\ \underline{then}\ tt\ \underline{else}\ (\underline{if}\ y\ \underline{then}\ tt\ \underline{else}\ ff)$,
$\wedge_{\mathcal{B}}, \vee_{\mathcal{B}} : \mathcal{B} \to \mathcal{B} \to \mathcal{B}$, *written infix,* $\wedge_{\mathcal{B}}$ *is the boolean conjunction,* $\vee_{\mathcal{B}}$ *the disjunction,*
$\neg_{\mathcal{B}} := \lambda x.\ \underline{if}\ x\ \underline{then}\ ff\ \underline{else}\ tt$,
$\neg_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$, *the boolean negation,*
$atom := \lambda x.\ \underline{if}\ x\ \underline{then}\ \top\ \underline{else} \bot$,
$atom : \mathcal{B} \to U$.
(atom transfers a boolean value to a formula, $atom(tt)$ *is a true formula — a type having one element, and* $atom(ff)$ *is a false formula — the type* $N_0 = \bot$, *that has as elimination rule the rule which corresponds to "ex falsum quodlibet")*

Note, that for decidable predicates p we have $\forall x, y \in N.atom(p(x,y)) \vee \neg(atom(p(x,y)))$.

The next task is to introduce certain functions and relations on N.

Definition 8.3 *(a)* $1 := S0$, $2 := SS0$, *etc.*

(b) $pred_N := \lambda x.P(x, 0, (u,v)u)$, $pred_N : N \to N$, *the predecessor on* N *that is* $pred_N 0 =_N 0$, $pred_N(Sx) =_N x$.

(c) $zero? := \lambda x.P(x, tt, (u,v)ff)$, $zero? : N \to \mathcal{B}$, *the test for* 0, *that is* $zero?0 =_{\mathcal{B}} tt$, $zero?(Sx) =_{\mathcal{B}} ff$,

(d) $<_{N,\mathcal{B}} := \lambda x.P(x, \lambda y.\neg_{\mathcal{B}}(zero?y),$
$(u,v)\lambda y.\ \underline{if}\,(zero?y)\ \underline{then}\ ff\ \underline{else}\ (v(pred_N y)))$,
$<_{N,\mathcal{B}}: N \to \mathcal{B}$, *the* $<$*-relation on* N*, which we write infix, and have:*
$(0 <_{N,\mathcal{B}} 0) =_{\mathcal{B}} ff$, $(0 <_{N,\mathcal{B}} St) =_{\mathcal{B}} tt$,
$(Sx <_{N,\mathcal{B}} 0) =_{\mathcal{B}} ff$, $(Sx <_{N,\mathcal{B}} Sy) =_{\mathcal{B}} (x <_{N,\mathcal{B}} y)$.
$<_N := \lambda x, y.atom(x <_{N,\mathcal{B}} y)$, *written infix, too.* $<_N: N \to U$ *is the* $<$*-relation, seen as a formula.*

(e) $\leq_{N,\mathcal{B}} := \lambda x.P(x, \lambda y.tt,$
$(u,v)\lambda y.\ \underline{if}\,(zero?y)\ \underline{then}\ ff\ \underline{else}\ (v(pred_N y)))$,
$<_{N,\mathcal{B}}: N \to \mathcal{B}$, *the leq-relation on* N *and we write* $\leq_{N,\mathcal{B}}$ *infix. We have:*
$(0 \leq_{N,\mathcal{B}} x) =_{\mathcal{B}} tt$,
$(Sx \leq_{N,\mathcal{B}} 0) =_{\mathcal{B}} ff$, $(Sx \leq_{N,\mathcal{B}} Sy) =_{\mathcal{B}} (x \leq_{N,\mathcal{B}} y)$.
$\leq_N := \lambda x, y.atom(x \leq_{N,\mathcal{B}} y)$, *written infix, too.*
$\leq_N: N \to U$.
We define $\forall x <_N t.\phi := \forall x \in N.x <_N t \to \phi$, *similarly for* $\exists$ *and* $\leq_N$.

(f) $=_{N,\mathcal{B}} := \lambda x.P(x, \lambda y.zero?y,$
$(u,v)\lambda y.\ \underline{if}\,(zero?y)\ \underline{then}\ ff\ \underline{else}\ (v(pred_N y)))$,
$=_{N,\mathcal{B}}: N \to \mathcal{B}$, *the decidable equality on* N*. We write* $=_{N,\mathcal{B}}$ *infix, and have:*
$(0 =_{N,\mathcal{B}} 0) =_{\mathcal{B}} tt$, $(0 =_{N,\mathcal{B}} St) =_{\mathcal{B}} ff$,
$(Sx =_{N,\mathcal{B}} 0) =_{\mathcal{B}} ff$, $(Sx =_{N,\mathcal{B}} Sy) =_{\mathcal{B}} (x =_{N,\mathcal{B}} y)$.
$\neq_{N,\mathcal{B}} := \lambda x, y.\neg_{\mathcal{B}} x =_{N,\mathcal{B}} y$.

(g) $+_N := \lambda x, y.P(y, x, (u,v)Sv)$,
$+_N : N \to N \to N$, *the addition on* N*. We write* $+_N$ *infix, and have:*
$x +_N 0 =_N x$, $x +_N Sy =_N S(x +_N y)$.

(h) $\dot{-} := \lambda x, y.P(y, x, (u,v)(pred_N v))$,
$\dot{-} : N \to N \to N$, *which is the minus-function on* N*, we write* $\dot{-}$ *infix, and have:*
$x \dot{-} 0 =_N x$, $x \dot{-} Sy =_N pred_N y(x \dot{-} y)$.

(i) $max_N := \lambda x, y.\ \underline{if}\ x <_N y\ \underline{then}\ y\ \underline{else}\ x$,
$min_N := \lambda x, y.\ \underline{if}\ x <_N y\ \underline{then}\ x\ \underline{else}\ y$,
$max_N, min_N : N \to N \to N$, *the maximum and minimum of two natural numbers and we write* $max_N\{a,b\}, min_N\{a,b\}$ *for* $max_N ab$, $min_N ab$.

(j) $\forall_{\mathcal{B},<} := \lambda n, \phi.P(n, tt, (u,v)\phi(u) \wedge_{\mathcal{B}} v)$,
$\forall_{\mathcal{B},<} : N \to (N \to \mathcal{B}) \to \mathcal{B}$, *that is*
$(\forall_{\mathcal{B},<} 0\phi) =_{\mathcal{B}} tt$,
$(\forall_{\mathcal{B},<}(Sn)\phi) =_{\mathcal{B}} (\phi n) \wedge_{\mathcal{B}} \forall_{\mathcal{B},<} n\phi)$.

$\exists_{\mathcal{B},<} := \lambda n, \phi.P(n, ff, (u, v)\phi(u) \vee_{\mathcal{B}} v)$, *that is*
$(\exists_{\mathcal{B},<} 0\phi) =_{\mathcal{B}} ff$,
$\exists_{\mathcal{B},<}(Sn)\phi =_{\mathcal{B}} ((\phi n) \vee_{\mathcal{B}} \exists_{\mathcal{B},<} n\phi)$.
Further $\forall_{\mathcal{B},\leq} := \lambda n, \phi.\forall_{\mathcal{B},<}(Sn)\phi$, $\exists_{\mathcal{B},\leq} := \lambda n, \phi.\exists_{\mathcal{B},<}(Sn)\phi$.

We write $\forall_{\mathcal{B}} x <_N n.\phi$ *for* $\forall_{\mathcal{B},<} n(\lambda x.\phi)$, $\forall_{\mathcal{B}} x \leq_N n.\phi$ *for* $\forall_{\mathcal{B},\leq} n(\lambda x.\phi)$, $\exists_{\mathcal{B}} x <_N n.\phi$ *for* $\exists_{\mathcal{B},<} n(\lambda x.\phi)$ *and* $\exists_{\mathcal{B}} x \leq_N n.\phi$ *for* $\exists_{\mathcal{B},\leq} n(\lambda x.\phi)$. $\forall_{\mathcal{B}}$, $\exists_{\mathcal{B}}$ *are used for bounded universal quantification over a decidable predicate, which can be decided (see 8.5 (e)).*

(k) *We write* $=$ *for* $=_N$.

We will now define lists as a pair $< n, f >$, where n is the length of the list, and $(f(i))_{i<n}$ are the elements of the list.

Definition 8.4 *Let* A *be a g-type, and assume (we will need this for stating the typing judgements),* $x, y, z, u, v, w \in Var_{ML}$, $ML_1^i W_R \vdash \Gamma \Rightarrow A'$ *type,* X, Y *variables that do not occur free or bounded in* Γ, A'.
$Alist := N \times (N \to A)$.
$lh := \lambda x.x0$, *and have* $\Gamma \Rightarrow lh : A'list \to A'$.
$(\cdot). := \lambda x, y.(x1)y$, *and have* $\Gamma \Rightarrow (\cdot). : A'list \to N \to A'$,
and write $(a)_i$ *for* $(\cdot).i$
$nil_y :=< 0, \lambda x.y >$, $\Gamma, y : A' \Rightarrow nil_y : A'list$, *(we omit the index* y*, if we have any usual dummy element, for instance* 0 *in case of* $A = N$*).*
$cons := \lambda x, y. < S(lh(y)), \lambda z.\ \underline{if}\ z = 0\ \underline{then}\ x\ \underline{else}\ (y)_{pred_N(z)} >$,
$\Gamma \Rightarrow cons : A' \to A'list \to A'list$,
$car := \lambda x.(x)_0$,
$\Gamma \Rightarrow car : A'list \to A'$
$cdr := \lambda x. < pred(lh(x)), \lambda y.(a)_{Sy} >$.
$cdr : A'list \to A'list$
$append := \lambda x, y. < lh(x) +_N lh(y), \lambda z\ \underline{if}\ z <_{N,\mathcal{B}} lh(x)\ \underline{then}\ (x)_z\ \underline{else}\ (y)_{lh(x)+_N z} >$.
$\Gamma \Rightarrow append : A'list \to A'list \to A'list$
$\cong_{Nlist,\mathcal{B}} := \lambda x, y.lh(x) =_{N,\mathcal{B}} lh(y) \wedge_{\mathcal{B}} \forall_{\mathcal{B}} z <_N lh(x).(x)_z =_{N,\mathcal{B}} (y)_z)$,
$\cong_{Nlist,\mathcal{B}}: Nlist \to Nlist \to \mathcal{B}$, *the equivalence of two lists, which we write infix, and*
$A \cong_{Alist} B := lh(B) = lh(C) \wedge \forall z <_N lh(B).(B)_z =_A (C)_z$.
(where z *is a variable, that does not occur free in* B *or* C*)*
(for $A : U$*, we can define a function* $\lambda X, Y.X \cong_{Alist} Y : Alist \to Alist \to U$ *and have then:*
$\Gamma, X, Y : Alist \Rightarrow X \cong_{Alist} Y$ *type)*
and for fixed n $< a_0, \ldots, a_n >_{List} :=< Sn, f >$, *where*
$f := \lambda x.\ \underline{if}\ x =_{N,\mathcal{B}} 0\ \underline{then}\ a_0\ \underline{else}$
$\underline{if}\ x =_{N,\mathcal{B}} 1\ \underline{then}\ a_1\ \underline{else} \cdots$
$\cdots \underline{if}\ x = n\ \underline{then}\ a_n\ \underline{else}\ a_0$,
and if x_i *are new variables, then* $ML_1^i W_R \vdash \Gamma, x_1 : A, \ldots, x_n : A \Rightarrow < x_1, \ldots, x_n >_{List}$: $A'list$
$Sublist\ x\ y$ *will be the list of the first* y *elements of the list* x:
$Sublist := \lambda x, y.\ \underline{if}\ y <_N lh(x)\ \underline{then} < y, x1 > \underline{else}\ x$,
$\Gamma \Rightarrow Sublist : A'list \to N \to A'list$, *we write* $Sublist(a, b)$ *for* $Sublist\ ab$.

We can easily prove the following properties:

Lemma 8.5 (a) $atom(tt)$, $\neg(atom(ff))$. *(which means: there exist g-terms* r, s*, such that* $ML_1^i W_R \vdash r : atom(tt)$, $ML_1^i W_R \vdash s : \neg(atom(ff))$*).*
$\forall x \in \mathcal{B}.atom(x) \vee \neg atom(x)$, *(which means that for all decidable predicates (functions with codomain* $\mathcal{B}$*) we have tertium non datur).*

(b) $\forall x, y \in \mathcal{B}.atom(x \wedge_{\mathcal{B}} y) \leftrightarrow atom(x) \wedge atom(y)$,
$\forall x, y \in \mathcal{B}.atom(x \vee_{\mathcal{B}} y) \leftrightarrow atom(x) \vee atom(y)$,
$\forall x \in \mathcal{B}.atom(\neg_{\mathcal{B}} x) \leftrightarrow \neg atom(x)$.

(c) $\forall x, y \in N.x <_N y \vee x = y \vee y <_N x$,
(which means: there exist a g-term r, such that
$ML_1^i W_R \vdash r : \forall x, y \in N.x <_N y \vee x = y \vee y <_N x$*)*,
$\forall x, y \in N.atom(x =_{N,\mathcal{B}} y) \leftrightarrow x = y$. $\forall x, y \in N.x <_N y \rightarrow (x \neq y \wedge \neg(y <_N x))$,
$\forall x, y \in N.x = y \rightarrow (\neg(x <_N y) \wedge \neg(y <_N x))$,
$\forall x, y \in N.x \leq_N y \leftrightarrow (x <_N y \vee x = y)$,

(d) $\forall x \in N.x \neq 0 \rightarrow x = S(pred_N(x))$.

(e) $\forall y \in N.\forall \phi \in N \rightarrow \mathcal{B}.atom(\forall_{\mathcal{B}} x <_N y.\phi) \leftrightarrow \forall x <_N y.atom(\phi y)$,
$\forall y \in N.\forall \phi \in N \rightarrow \mathcal{B}.atom(\exists_{\mathcal{B}} x <_N y.\phi) \leftrightarrow \exists x <_N y.atom(\phi y)$,
similarly for $\leq_N$.

(f) *Let* $\Gamma \Rightarrow A : type$, $\Gamma \Rightarrow d : A$.
$\forall x, y \in Nlist.atom(x \cong_{Nlist,\mathcal{B}} y) \leftrightarrow x \cong_{Nlist} y$.
$\forall x \in Alist.x \cong_{Alist} nil_d \leftrightarrow lh(x) = 0$.
$\Gamma \Rightarrow \forall x \in Alist.\neg(x \cong_{Alist} nil_d) \rightarrow x \cong_{Alist} cons(car(x), cdr(x))$.
$\Gamma \Rightarrow \forall x \in Alist.append(nil_d, x) \cong_{Alist} x$,
$\Gamma \Rightarrow \forall x, y \in Alist, z \in A.append(cons(z, x), y) \cong_{Alist} cons(z, append(x, y))$

Proof:
(a) is obvious, (b) follows by boolean induction. In the second assertion of (c), "$\rightarrow$" follows by induction on $x, y : N$, for "$\leftarrow$" we use again the same induction, and argue, if $0 = Sy$, then $0_1 \in N_1 = P(Sy, N_0, (u, v)N_1) = P(0, N_0, (u, v)N_1) = N_0 = atom(0 =_{N,\mathcal{B}} Sy)$, and if $Sx = Sy$, then $x = pred_N(Sx) = pred_N(Sy) = y$. The other assertions in (c), (d), (e) follow by induction on N.
In (f), the first assertion follows (b) and (e), by induction on the length of the lists, the other one follow easily.

The most complicated definition is to introduce the analogue of the subsets of the natural numbers. We will use in this proof four different possibilities for introducing the power set of N, which we present in increasing order of their complexity: The finite sets $\mathcal{P}^{fin}(N)$, defined as lists of natural numbers, the decidable sets $\mathcal{P}^{dec}(N)$, defined as boolean functions on N, which decide, which elements belong to the set, the ordinary power set $\mathcal{P}(N)$, defined as a functions $f : N \rightarrow U$, where the elements should be those $n : N$, such that fn is not an empty type, and the subclasses of N $\mathcal{C}l_y(N)$, which are types (seen as properties) with free variable y.
The subsets with lower complexity have, roughly speaking, the advantage of better degree of decidability, whereas we need the subsets of high complexity, to introduce concepts which correspond to the highly impredicative definitions, possible in the corresponding systems of analysis.
For each power set we will introduce the relation η, which stands for the element relation (for the first two power sets additionally the decidable element relation $\eta_{\mathcal{B}}$), the types $\forall x\ \eta\ A.B$, (with intended meaning for all x in A we have B), $\exists x\ \eta\ A.B$, the relations the relations $\subset$ for subset and $\cong$ for "have the same elements" between power sets, union ($\cup$), intersection ($\cap$) and difference ($\setminus$). Further we lift an element of one power set to the next complicated power set (operations $\cdot^{+}$, $\cdot^{\mathcal{P}}$, $\cdot^{+,\mathcal{C}l,y}$, and we define the finite sets

$\{a_0, \dots, a_n\}$, the empty set $\emptyset$, and, except in the finite case the set corresponding to N, namely N_{dec}, $N_{\mathcal{P}}$, $N_{Cl,y}$.
We will give, whenever possible, closed terms, which we can do, as long as the types are constructed of elements of U or elements of some fixed type related to U. For the case that the types are no longer elements of this type, we need another definition, which is for each type a new syntactical object.
We will use capital letters like A, B, C and X, Y, Z to indicate elements of the four power sets of N.

Definition 8.6 *(a) The four kinds of subsets are:*

The finite subsets are the elements of $\mathcal{P}^{fin}(N) := Nlist$,
the decidable subsets are those of $\mathcal{P}^{dec}(N) := N \to \mathcal{B}$,
the ordinary subsets are the elements of $\mathcal{P}(N) := N \to U$.
For the subclasses of N *we have no type containing all classes, but we have only the following meta statement:*
$\Gamma \Rightarrow A \in \mathcal{C}l_y(N)$ *is defined as* $ML_1^i W_R \vdash \Gamma, y : N \Rightarrow A$ *type.*

The finite subsets can be seen as a list of the elements of the set, the decidable subsets are functions of boolean value, which decide for each element, if it belongs to the set. The ordinary subsets are functions, that assign an element of U *to each natural number. If we see this element as a formula, the set is supposed to contain all elements, for which this formula is true (although this sentence is just some heuristic, we cannot say in type theory that an element of* U *is true, only that we have some* $t : U$*). To exhaust the full proof theoretical strength of Martin-Löf's type theory we need classes, which correspond to classes in systems of the analysis. A class is a type, depending on one free variable* y *(and eventually on a bigger context), and the class* $A \in \mathcal{C}l_y(N)$ *stands for* $\{x \in \mathbb{N} | A[y/x]\}$*. When using classes, we have to be careful with the use of free and bounded variables.*

(b) In $\mathcal{P}^{fin}(N)$ *and* $\mathcal{P}^{dec}(N)$ *we have decidable element relations* $\eta_{fin,\mathcal{B}}, \eta_{dec,\mathcal{B}}$, *which distinguish these from* $\mathcal{P}(N)$, $\mathcal{C}l_y(N)$*:*

$\eta_{fin,\mathcal{B}} := \lambda y, X.\exists_{\mathcal{B}} x <_N lh(X).y =_{N,\mathcal{B}} (X)_x$, $\eta_{fin,\mathcal{B}}: N \to \mathcal{P}^{fin}(N) \to \mathcal{B}$;

$\eta_{dec,\mathcal{B}} := \lambda y, X.Xy$, $\eta_{dec,\mathcal{B}}: N \to \mathcal{P}^{dec}(N) \to \mathcal{B}$.

Both relations are written infix.

(c) The element relations η_{fin}, η_{dec}, η *and* $\eta_{Cl,y}$ *are defined as follows:*

$\eta_{fin} := \lambda y, X.atom(y\ \eta_{fin,\mathcal{B}}\ X)$, $\eta_{fin}: N \to \mathcal{P}^{fin}(N) \to U$;

$\eta_{dec} := \lambda y.X.atom(y\ \eta_{dec,\mathcal{B}}\ X)$, $\eta_{dec}: N \to \mathcal{P}^{dec}(N) \to U$;

$\eta := \lambda y, X.Xy$, $\eta: N \to \mathcal{P}(N) \to U$.

all these three relations are written infix, and $y\ \eta.\ X$ *can be seen heuristically as a formula.*

$t\ \eta_{Cl,y}\ B := B'[y/t]$, *where* $B' =_\alpha B$ *such that* $B'[y/t]$ *is an allowed substitution (we can do this for all the finitely many types occurring in the proof we construct)*

We further define
$\not\eta_{fin} := \lambda y, X.\neg y\ \eta_{fin}\ X$, $\not\eta_{fin}: N \to \mathcal{P}^{fin}(N) \to U$,
similarly $\not\eta_{dec}$, $\not\eta$, *and*
$t\ \not\eta_{Cl,y}\ B := \neg(t\ \eta_{Cl,y}\ B)$.

(d) $\mathcal{P}^{fin}(N)$ *is distinguished from the other kinds of subsets, by having decidable quantification over boolean predicates:*

$\forall_{\mathcal{B}} := \lambda X, f.\forall_{\mathcal{B}} x <_N lh(X).f((X)_x),$
$\exists_{\mathcal{B}} := \lambda X, f.\exists_{\mathcal{B}} x <_N lh(X).\phi((X)_x).$
$\forall_{\mathcal{B}}, \exists_{\mathcal{B}} : \mathcal{P}^{fin}(N) \to (N \to \mathcal{B}) \to \mathcal{B},$

and we write $\forall_{\mathcal{B}} x \ \eta \ B.\phi$ *for* $\forall_{\mathcal{B}} B(\lambda x.\phi)$ *and* $\exists_{\mathcal{B}} x \ \eta \ B.\phi$ *for* $\exists_{\mathcal{B}} b(\lambda x.\phi)$.

We can prove:
$\forall X \in \mathcal{P}^{fin}(N).\forall f \in X \to \mathcal{B}.atom(\forall_{\mathcal{B}} x \ \eta \ X.f) \leftrightarrow \forall x \in N.x \ \eta_{fin} \ X \to atom(fx),$
$\forall X \in \mathcal{P}^{fin}(N).\forall f \in X \to \mathcal{B}.atom(\exists_{\mathcal{B}} x \ \eta \ X.f) \leftrightarrow \exists x \in N.x \ \eta_{fin} \ X \wedge atom(fx).$

(e) *We define the quantification over subsets of* N *first, for functions with values in* U *as closed terms:*

$\Sigma_{fin} := \lambda X, Y.\Sigma x \in N.x \ \eta_{fin} \ X \wedge (Yx),$
$\Pi_{fin} := \lambda X, Y.\Pi x \in N.x \ \eta_{fin} \ X \to (Yx),$
$\Sigma_{fin}, \Pi_{fin} : \mathcal{P}^{fin}(N) \to (N \to U) \to U;$

$\Sigma_{dec} := \lambda X, Y.\Sigma x \in N.x \ \eta_{dec} \ X \wedge (Yx),$
$\Pi_{dec} := \lambda X, Y.\Pi x \in N.x \ \eta_{dec} \ X \to (Yx),$
$\Sigma_{dec}, \Pi_{dec} : \mathcal{P}^{dec}(N) \to (N \to U) \to U;$

$\Sigma_{\mathcal{P}} := \lambda X, Y.\Sigma x \in N.\Sigma y \in (x \ \eta \ X).(Yxy),$
$\Pi_{\mathcal{P}} := \lambda X, Y.\Pi x \in N.\Pi y \in (x \ \eta \ X).(Yxy),$
$\Sigma_{\mathcal{P}}, \Pi_{\mathcal{P}} : \Pi X \in \mathcal{P}(N).(\Pi x \in N.x \ \eta \ X \to U) \to U.$

We write $\forall$ *for* Π, $\exists$ *for* Σ, *(when our heuristic is to speak of formulas rather than functions), and write* $\Sigma_{\mathcal{B}} x \ \eta \ B.\phi$ *for* $\Sigma_{\mathcal{B}} B(\lambda x.\phi)$, *similarly for* Π, $\forall$, $\exists$, *and for indices dec, and in the case of ordinary subsets* $\Sigma x \ \eta \ A.B$ *or* $\exists x \ \eta \ A.B$ *for* $\Sigma_{\mathcal{P}} A.(\lambda x, y.B)$, $\forall x \ \eta \ A.B$ *or* $\Pi x \ \eta \ A.B$ *for* $\Pi_{\mathcal{P}} A(\lambda x, y.B)$, *omitting the index* y *if the variable* y *does not occur free in* B *(we assume that in this case we choose a new variable* y*).*

(f) *If we have no function with value in* U, *but just a type with a free variable, quantification is defined as follows (where* x, z *are always new variables):*

$\Sigma x \ \eta_{fin,typ} \ A.B := \Sigma x \in N.x \ \eta_{fin} \ A \wedge B$, $\Pi x \ \eta_{fin,typ} \ A.B := \Pi x \in N.x \ \eta_{fin} \ A \to B;$

$\Sigma x \ \eta_{dec,typ} \ A.B := \Sigma x \in N.x \ \eta_{dec} \ A \wedge B$, $\Pi x \ \eta_{dec,typ} \ A.B := \Pi x \in N.x \ \eta_{dec} \ A \to B;$

$\Sigma x \ \eta_{typ} \ A.B := \Sigma x \in N.x \ \eta \ A \wedge B,$
$\Pi x \ \eta_{typ} \ A.B := \Pi x \in N.x \ \eta \ A \to B.$

Since we have $A : \mathcal{P}(N), B : N \to U \Rightarrow r : \Pi x \ \eta \ A.B \leftrightarrow \Pi x \ \eta_{typ} \ A.B$, *the same for* Σ *and in the cases* fin *and* dec, *we will omit the index typ, if there is no confusion.*

$\Sigma x \ \eta_{Cl,y} \ A.B := \Sigma x \in N.x \ \eta_{Cl,y} \ A \wedge B$
$\Pi x \ \eta_{Cl,y} \ A.B := \Pi x \in N.x \ \eta_{Cl,y} \ A \to B$

(g) *For the finite subsets we have a decidable subset relation:*

$\subset_{fin,\mathcal{B}} := \lambda X, Y.\forall_{\mathcal{B}} x \ \eta \ X.x \ \eta_{fin,\mathcal{B}} \ Y$, $\subset_{fin,\mathcal{B}} : \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N) \to \mathcal{B}$, *written infix.*

For all sets we define the subset relation by:

$\subset_{fin} := \lambda X, Y.atom(X \subset_{fin,\mathcal{B}} Y)$, $\subset_{fin} : \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N) \to U;$

$\subset_{dec} := \lambda X, Y.\forall x \in N.x \ \eta_{dec} \ X \to x \ \eta_{dec} \ Y,$
$\subset_{dec} : \mathcal{P}^{dec}(N) \to \mathcal{P}^{dec}(N) \to U;$

$\subset := \lambda X, Y.\forall x \in N.x \ \eta \ X \to x \ \eta \ Y,$
$\subset : \mathcal{P}(N) \to \mathcal{P}(N) \to U;$

We write all these relations infix.

$B \subset_{Cl,y} C := \forall x \in N.x\ \eta_{Cl,y}\ B \to x\ \eta_{Cl,y}\ C$,
where x *is a new variable.*
If $\Gamma \Rightarrow B, C \in \mathcal{C}l_y(N)$, *then* $\Gamma \Rightarrow B \subset_{Cl,y} C$ *type.*

(h) *For the finite subsets we have a decidable relation* $\cong$, *for "have the same elements":* $\cong_{fin,\mathcal{B}} := \lambda X, Y.X \subset_{fin,\mathcal{B}} Y \wedge_{\mathcal{B}} Y \subset_{fin,\mathcal{B}} X$,
$\cong_{fin,\mathcal{B}}: \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N) \to \mathcal{B}$, *written infix.*

For all subsets we define $\cong$ *as follows:*

$\cong_{fin} := \lambda X, Y.atom(X \cong_{fin,\mathcal{B}} Y)$, $\cong_{fin}: \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N) \to U$;

$\cong_{dec} := \lambda X, Y.X \subset_{dec} Y \wedge Y \subset_{dec} X$, $\cong_{dec}: \mathcal{P}^{dec}(N) \to \mathcal{P}^{dec}(N) \to U$;

$\cong := \lambda X, Y.X \subset Y \wedge Y \subset X$, $\cong: \mathcal{P}(N) \to \mathcal{P}(N) \to U$;

We write all these relations infix.

$B \cong_{Cl,y} C := B \subset_{Cl,y} C \wedge C \subset_{Cl,y} B$.
If $\Gamma \Rightarrow B, C \in \mathcal{C}l_y(N)$, *then* $\Gamma \Rightarrow B \cong_{Cl,y} C$ *type,*

(i) *We define union* $\cup$, *intersection* $\cap$ *and set difference* $\setminus$ *as follows:*
$\cup_{fin} := \lambda X, Y. < lh(X) +_N lh(Y), \lambda x.\ \underline{if}\ (x <_{N,\mathcal{B}} lh(X))\ \underline{then}\ ((X)_x)\ \underline{else}\ (Y_{x \dot{-} lh(X)}) >$,
$\cap_{fin} := \lambda X, Y.P(lh(X), \emptyset_{fin}, (u, v)\ \underline{if}\ ((X)_u)\ \eta_{fin,\mathcal{B}}\ Y\ \underline{then}$
$(v \cup_{fin} \{(X)_u\})\ \underline{else}\ v)$,
$\setminus_{fin} := \lambda X, Y.P(lh(X), \emptyset_{fin}, (u, v)\ \underline{if}\ ((X)_u)\ \eta_{fin,\mathcal{B}}\ Y\ \underline{then}\ v\ \underline{else}\ (v \cup_{fin} \{(X)_u\}))$,
$\cup_{fin} \cap_{fin}, \setminus_{fin}: \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N) \to \mathcal{P}^{fin}(N)$;

$\cap_{dec} := \lambda X, Y.\lambda y.y\ \eta_{dec,\mathcal{B}}\ X \wedge_{\mathcal{B}} y\ \eta_{dec,\mathcal{B}}\ Y$,
$\cup_{dec} := \lambda X, Y.\lambda y.y\ \eta_{dec,\mathcal{B}}\ X \vee_{\mathcal{B}} y\ \eta_{dec,\mathcal{B}}\ Y$,
$\setminus_{dec} := \lambda X, Y.\lambda y.y\ \eta_{dec,\mathcal{B}}\ X \wedge_{\mathcal{B}} \neg_{\mathcal{B}}(y\ \eta_{dec,\mathcal{B}}\ Y)$,
$\cap_{dec}, \cup_{dec}, \setminus_{dec} : \mathcal{P}^{dec}(N) \to \mathcal{P}^{dec}(N) \to \mathcal{P}^{dec}(N)$;

$\cap := \lambda X, Y.\lambda y.y\ \eta\ X \wedge y\ \eta_{dec}\ Y$,
$\cup := \lambda X, Y.\lambda y.y\ \eta\ X \vee y\ \eta\ Y$,
$\setminus := \lambda X, Y.\lambda y.y\ \eta\ X \wedge \neg(y\ \eta\ Y)$,
$\cap, \cup, \setminus : \mathcal{P}(N) \to \mathcal{P}(N) \to \mathcal{P}(N)$;

all these functions are written infix.

$B \cap_{Cl,y} C := B \wedge C$,
$B \cup_{Cl,y} C := B \vee C$,
$B \setminus_{Cl,y} C := B \wedge \neg(C)$.
If $\Gamma \Rightarrow B, C \in \mathcal{C}l_y(N)$, *then*
$\Gamma \Rightarrow B \cup_{Cl,y} C, B \cap_{Cl,y} C, B \setminus_{Cl,y} C \in \mathcal{C}l_y(N)$.

We can easily prove
$\forall X, Y \in \mathcal{P}^{fin}(N).\forall x \in N.x\ \eta_{fin}\ X \cup_{fin} Y \leftrightarrow (x\ \eta_{fin}\ X \vee x\ \eta_{fin}\ Y)$,
$\forall X, Y \in \mathcal{P}^{fin}(N).\forall x \in N.x\ \eta_{fin}\ X \cap_{fin} Y \leftrightarrow (x\ \eta_{fin}\ X \wedge x\ \eta_{fin}\ Y)$,
$\forall X, Y \in \mathcal{P}^{fin}(N).\forall x \in N.x\ \eta_{fin}\ X \setminus_{fin} Y \leftrightarrow (x\ \eta_{fin}\ X \wedge x\ \not\eta_{fin}\ Y)$,
similarly for $\mathcal{P}^{dec}(N)$, $\mathcal{P}(N)$, *and for the classes we prove, that, if* $\Gamma \Rightarrow A \in \mathcal{C}l_y(N)$, $\Gamma \Rightarrow B \in \mathcal{C}l_y(N)$, *then* $\Gamma \Rightarrow \forall x \in N.x\ \eta_{Cl,y}\ A \cup_{Cl,y} B \leftrightarrow (x\ \eta_{Cl,y}\ A \vee x\ \eta_{Cl,y}\ B)$,
similarly for the other functions.

(j) *We define the lifting from* $\mathcal{P}^{fin}(N)$ *to* $\mathcal{P}^{dec}(N)$, *from* $\mathcal{P}^{dec}(N)$ *to* $\mathcal{P}(N)$ *and from* $\mathcal{P}(N)$ *to* $\mathcal{C}l_y(N)$:

$\cdot^{+} := \lambda Y.\lambda y.y\ \eta_{fin,\mathcal{B}}\ Y$, $\cdot^{+} : \mathcal{P}^{fin}(N) \to \mathcal{P}^{dec}(N)$, *and we write* B^{+} *for* $\cdot^{+}B$;

$\cdot^{\mathcal{P}} := \lambda X.\lambda y.y\ \eta_{dec}\ X$, $\cdot^{\mathcal{P}} : \mathcal{P}^{dec}(N) \to \mathcal{P}(N)$, *written* $B^{\mathcal{P}}$ *for* $\cdot^{\mathcal{P}}B$;

$\cdot^{+,Cl,\cdot} := \lambda y.\lambda B.y\ \eta\ B$, *written* $B^{+,Cl,y}$ *for* $\cdot^{+,Cl,\cdot}yB$;

and we have $B : \mathcal{P}(N) \Rightarrow B^{+,Cl,y} \in \mathcal{C}l_y(N)$.

We see easily, that
$\forall X \in \mathcal{P}^{fin}(N).\forall x \in N.x\ \eta_{fin,\mathcal{B}}\ X = x\ \eta_{dec,\mathcal{B}}\ X^{+}$, $\forall X,Y \in \mathcal{P}^{fin}(N).(X \cup_{fin} Y)^{+} \cong_{dec} (X^{+} \cup_{dec} Y^{+})$ $\forall X \in \mathcal{P}^{dec}(N).\forall Y \in N \to U.(\forall_{\mathcal{P}} x \in X^{\mathcal{P}}.(Yx)) \leftrightarrow (\forall_{dec} x \in X.(Yx))$ *etc. similarly for* $\cdot^{\mathcal{P}}$, $\cdot^{+,Cl,y}$, *therefore we will, if there is no confusion, omit these superscripts, and subscripts* dec, fin, $\mathcal{P}$, Cl, y.

(k) *We define the empty set:*

$\emptyset_{fin} := nil$,
$\emptyset_{dec} := \emptyset_{fin}^{+}$,
$\emptyset := \emptyset_{dec}^{\mathcal{P}}$,
$\emptyset_{Cl,y} := \emptyset^{+,Cl,y}$.

We define the finite sets:

$\{a_0,\ldots,a_n\}_{fin} :=< a_0,\ldots,a_n >_{List}$,
$\lambda x_0,\ldots,x_n.\{x_0,\ldots,x_n\}_{fin} : N \to N \to \cdots N \to \mathcal{P}^{fin}(N)$;

$\{a_0,\ldots,a_n\}_{dec} := \{a_0,\ldots,a_n\}_{fin}^{+}$;

$\{a_0,\ldots,a_n\} := \{a_0,\ldots,a_n\}_{dec}^{\mathcal{P}}$;

$\{a_0,\ldots,a_n\}_{Cl,y} := \{a_0,\ldots,a_n\}^{+,Cl,y}$.

The set of natural numbers can be represented as follows:

$N_{dec} := \lambda x.tt$, $N_{dec} : \mathcal{P}^{dec}(N)$;
$N_{\mathcal{P}} := N_{dec}^{\mathcal{P}}$;
$N_{Cl,y} := N^{+,Cl,y}$.

Remark 8.7 *In the following, we will often have statements, which can be stated for all elements of* $\mathcal{P}(N)$ *and for all classes, and have similar proofs. So we have the following convention:*
The statement "for $Y : \mathcal{P}(N)$ *or* $Y \in \mathcal{C}l_y(N)$ *holds* ϕ*" stands for*
"$ML_1^iW_R \vdash \forall Y \in \mathcal{P}(N).\phi$ *and, if* $ML_1^iW_R \vdash \Gamma, y \in N \Rightarrow Y$ *type, then* $ML_1^iW_R \vdash \Gamma \Rightarrow \phi'$*", where in* ϕ' *we replace* η *by* $\eta_{Cl,y}$*, and rename all bounded variables, such that they are different from all variables in* Γ*, similarly for the other operations on classes.*
The statement "for $X, Y : \mathcal{P}(N)$ *or* $X, Y \in \mathcal{C}l_y(N)$ *stands for "for* $X : \mathcal{P}(N)$ *or* $X \in \mathcal{C}l_y(N)$*. for* $Y : \mathcal{P}(N)$ *or* $Y \in \mathcal{C}l_z(N)$ *we have* ϕ*", that is, unfolding it we have four statements for* $X, Y : \mathcal{P}(N)$*,* $X \in \mathcal{C}l_y(N), Y : \mathcal{P}(N)$*,* $Y \in \mathcal{C}l_z(N), X : \mathcal{P}(N)$ *and* $X \in \mathcal{C}l_y(N), Y \in \mathcal{C}l_z(N)$.

The next task is to define the subtree ordering $\prec$ on the W-type. An element $sup(a,f)$ of a type $Wx \in A.B$ is a tree, having immediate subtrees $(fz)_{z\in A}$. By iterating the step to the immediate subtree, we get all subtrees. So we define: $\alpha \prec \beta$ iff we can get from β to α by always going to an immediate subtree.
We need two definitions, one where we quantify over all $A : U$ and $B : A \to U$, and another definition, suitable for $A : type$ and $x : A \Rightarrow Btype$, so we have statements only for one special type A and B.

Definition 8.8 *(a) Assume A, B g-types, α, β g-terms, x a variable.*
The immediate subtree relation for arbitrary trees:
$\alpha \prec^1_{A,x,B} \beta := \exists x \in A.\exists f \in B \to (Wx \in A.B).\exists z \in B.$
$\beta =_{Wx\in A.B} sup(x, f) \wedge \alpha =_{Wx\in A.B} fz$
(where f, z are new variables)
The subtree relation on arbitrary trees:
$\alpha \prec_{A,x,B} \beta :=$
$\exists n \in N.0 <_N n \wedge \exists f \in (N \to (Wx \in A.B)).$
$(f0) =_{Wx\in A.B} \beta \wedge (fn) =_{Wx\in A.B} \alpha \wedge$
$\forall i <_N n.(fi) \prec^1_{A,x,B} (f(Si)).$
(where n, i, f are new variables).

$\alpha \preceq_{A,x,B} \beta := \alpha \prec_{A,x,B} \beta \vee \alpha =_{Wx\in A.B} \beta.$
We have, if $ML^i_1W_R \vdash \Gamma \Rightarrow A$ type, $ML^i_1W_R \vdash \Gamma, x : A \Rightarrow B$ type, α, β are new variables, then

$$ML^i_1W_R \vdash \Gamma \Rightarrow \forall\alpha, \beta \in Wx \in A.B. \Rightarrow \alpha \prec_{A,B} \beta \text{ type}$$

the same for $\preceq_{A,B}$.

(b) The immediate subtree relation for trees "in U":
$\prec^1_{Univ} := \lambda A, B.\lambda\alpha, \beta.\exists x \in A.\exists f \in (Bx) \to (Wx \in A.(Bx)).\exists z \in (Bx).$
$\beta =_{Wx\in A.(Bx)} sup(x, f) \wedge \alpha =_{Wx\in A.(Bx)} fz,$
(where $A, B, \alpha, \beta, x, f$ are different variables)

$$ML^i_1W_R \vdash \prec^1_{Univ}: \Pi A \in U.\Pi B \in A \to U.Wx \in A.(Bx) \to Wx \in A(Bx) \to U$$

We write $\alpha \prec^1_{Univ,A,B} \beta$ for $\prec_{Univ} AB\alpha\beta$.
The subtree relation on these trees:
$\prec_{Univ} := \lambda A.\lambda B.\lambda\alpha, \beta.$
$\exists n \in N.0 <_N n \wedge \exists f \in N \to (Wx \in A.(Bx)).$
$(f0) =_{Wx\in A.(Bx)} \alpha \wedge (fn) =_{Wx\in A.(Bx)} \beta \wedge$
$\forall i <_N k.(fi) \prec^1_{Univ,A,B} (f(Si)).$
Further:
$\preceq_{Univ} := \lambda A.\lambda B.\lambda\alpha, \beta.\alpha \prec \beta \vee \alpha =_{Wx\in A.(Bx)} \beta.$

$$ML^i_1W_R \vdash \prec_{Univ}: \Pi A \in U.\Pi B \in A \to U.Wx \in A.(Bx) \to Wx \in A.(Bx) \to U$$

the same for $\preceq_{Univ}$ and again write $\alpha \prec_{Univ,A,B} \beta$. We will usually omit the Index Univ or even A, B, so we write $\alpha \prec \beta$ for $\prec_{Univ} AB\alpha\beta$ if this does not cause any confusion.

Lemma 8.9 *Assume $\Gamma \Rightarrow A$ type, $\Gamma, x \in A \Rightarrow B$ type. Then:*

(a) $\forall\alpha, \beta, \gamma \in (Wx \in A.B).\alpha \preceq \beta \leftrightarrow (\alpha \prec \beta \vee \alpha =_{Wx\in A.B} \beta)$

(b) $\forall\alpha, \beta, \gamma \in (Wx \in A.B).((\alpha \prec \beta \wedge \beta \preceq \gamma) \to \alpha \prec \gamma) \wedge ((\alpha \preceq \beta \wedge \beta \prec \gamma) \to \alpha \prec \gamma)$

(c) $\forall\alpha \in (Wx \in A.B).\neg(\alpha \prec \alpha).$

(d) $\forall\alpha \in Wx \in A.B.\forall x \in A.\forall s \in (B \to Wx \in A.B).$
$\alpha \prec sup(r, s) \leftrightarrow (\exists y \in B.\alpha \preceq sy).$

(e) (a) - (d) are valid, if we quantify over all $X : U$, $Y : X \to U$, e.g. in (a):
$\forall X \in U.\forall Y \in (X \to U).\forall\alpha, \beta, \gamma \in (Wx \in A.(Bx)).$
$(\alpha \preceq \beta \leftrightarrow \alpha \prec \beta \vee \alpha =_{Wx\in A.(Bx)} \beta)$

Proof: (a), (b), (d) are trivial.
(c) We prove by Induction on $\alpha : Wx \in A.B. \neg(\alpha \prec \alpha)$:
Assume f, n as in the Definition.
We have $0 <_N n$, $f0 =_{Wx \in A.B} sup(r,s) \wedge f1 =_{Wx \in A.B} sy$ for some r, s, y. Define $g := \lambda x.$ <u>if</u> $x <_{N,\mathcal{B}} n$ <u>then</u> $f(S(x))$ <u>else</u> sy, $g : N \to Wx \in Ay.B$. Then using g we conclude $sx \prec sx$, a contradiction to I.H.

We will now explain the way of carrying out the well-ordering proof.
A first attempt to do this, is to define operations like $+$, $\omega^{\cdot}$, $\Omega.$, and the collapsing function $\lambda x, y.D_x y$ on some huge W-type $Wx \in A.B$, and then map some ordinal denotation system OT on this W-type, using the recursion on $Wx \in A.B$ to prove transfinite induction on OT. This can be done, as long as we restrict ourselves to ordinals of O_n, the n-th number class. For bigger number classes, we need for the definition of $D_x y$ the relation $\tau(y) \prec x$: we need some relation between the elements of a tree and its branching type. When we think of them as elements of two trees, and of $\prec$ as the subtree ordering (modulo isomorphisms between trees), we have the problem, that this ordering is not decidable, we can not define a function $D_x y$.

The next idea is, to use the decidable ordering $\prec$ on OT, together with fundamental sequences $(a[x])_{x \prec \tau(a)}$, where $\tau(a)$ is the type of the ordinal a, and besides some properties we have for limes ordinals $a = sup\{a[x] | x \prec \tau(a)\}$. The ordinals $a[x]$ correspond to immediate subtrees, we had before. Let $W_1 := Wx \in N.(\Sigma y \in N.y \prec \tau(a))$. If $sup(r,s) \in W_1$, then r should be the label, an ordinal, of this tree. (We will use arbitrary labels in N for technical reasons) Then we distinguish those trees of W_1, where the fundamental sequences of the labels of its subtrees correspond to the ordering in W_1, that is we define $Correct(\alpha)$, (in this informal part we omit some bounds for quantifiers, so that the formula can be easier read)

$$Cor(\alpha) := \forall r, s.sup(r,s) \preceq \alpha \to \forall x \in (\Sigma y \in N.y \; \eta \; \widehat{\tau(r)}.\exists s'.sx = sup(r[x0], s')$$

(where $\widehat{\tau(r)}$ is some set related to $\tau(r)$.)
Now to define the functions we need, we need some well-ordering on $\{x \in OT | x \prec \tau(a)\}$, which we originally want to prove by this method. The idea which helps, which is the first step towards the "ausgezeichnete Mengen", is to replace $\tau(x)$ by $\tau(x) \cap X$, where we assume, that we know the well-ordering of X.
Actually we will use $\tau(x)^X := (\tau(x)^{\prec} \cap X) \cup \tau'(x)$ where $\tau'(x)$ contains some elements of $\tau(x)^{\prec}$, which we want to include in any case. Now we define W_1, $Cor(\alpha)$ as before and let $W(X)$ be the natural numbers, that are labels of correct trees of W_1.

The last problem is, that now $\tau(x)^X$ occurs negatively in the definition of $W(X)$, that is, the bigger X, the smaller $W(X)$.
To get sets, which contain Ω_α for finite α or α in some ordering, which is already proven to be a well-ordering, we could construct $O_0 := W(\emptyset) \cap \Omega_1$, $O_1 := W(O_0) \cap \Omega_2$ and so on, so that at every step we know in some sense, that O_n is complete. To get sets which contain Ω_{Ω_1} and stronger cardinals and give very strong well ordering proofs, Buchholz has introduced the concept of the "ausgezeichnete Mengen", the distinguished sets, which is the property, which all the O_α we constructed before have in common, and which we needed. The property is some sort of stability in the sense that the segment formed by the set does not grow by forming $W(X)$. We define X is a "ausgezeichnete Menge" ($Ag(X)$), if $X \sqsubseteq W(X)$, that is: $\forall x \; \eta \; X.X|x \cong W(X)|x$
Now these sets allow induction (since we have a corresponding tree $\alpha \in W_1(X)$) and we can define all the functions we need.

To get the full power of Martin-Löf's type theory, in a next step we built “ausgezeichnete Klassen”. First we built the union of all “ausgezeichnete Mengen”, which is a class, closed under Ω.. Then we built $\mathcal{W}_0 := \mathcal{W} \cap I$, $\mathcal{W}_{i+1} := W(\mathcal{W}_i) \cap \Omega_{I+i+1}$, which are “ausgezeichnete Klassen”, such that $I \;\eta\; \mathcal{W}_1$, $\Omega_{I+i} \;\eta\; \mathcal{W}_{i+1}$.

Historically, these steps are not the way, the author found this attempt. They are only a way to motivate it. Actually, after trying the first attempt, the author tried to adopt the paper of Buchholz [Buc90], the until now clearest version of the method of the “ausgezeichnete Mengen”, to Martin-Löf's type theory. The method of “ausgezeichnete Mengen” goes back to Buchholz ([Buc75a]), who first needed $\Pi^1_2 - CA$ for introducing them and where further developed by Schütte, who discovered, how to define these concepts in weaker subsystems of analysis.

But before introducing all these concepts, we need some ordinal denotation system OT. We will introduce this in the next chapter 9, here we only assume, that we have a system OT, with certain subsets an ordering $\prec$, functions like $+$, that has certain properties, is given:

General Assumption 8.10 *We assume an ordinal denotation system*

$$OT : \mathcal{P}^{dec}(N),$$

which we will introduce in chapter 9, together with elements

$$0_{OT}, 1_{OT}, \omega, I \;\eta\; OT$$

(I should be a representative for the first weakly inaccessible cardinal or its recursive analogue) sets

$$Lim, A, R : \mathcal{P}^{dec}(N)$$

(where Lim will be the denotations for limit ordinals, A the additive principal numbers and R the regular cardinals) such that

$$R \subset A \subset Lim \subset OT, \;\; I \;\eta\; R,$$

and the following functions:

$$\prec_{OT}, \preceq_{OT} : N \to N \to U$$

(written in Infix, we will, if there is no confusion, omit the index OT),
We further assume

$$\tau : N \to N$$

such that $\forall x \;\eta\; OT.\tau(x) \;\eta\; R \cup \{0, 1_{OT}, \omega\}$,

$$\cdot[\cdot] : N \to N \to N$$

(written $a[b]$ *for* $[\cdot] \cdot ab$*)*

$$+ : N \to N \to N$$

(written infix)

$$NF_{+,\mathcal{B}} : N \to N \to \mathcal{B}$$

(for + normal form), we write $NF_{+,\mathcal{B}}(x, y)$ *for* $NF_{+,\mathcal{B}}xy$ *and define*

$$NF_+ := \lambda x, y.atom(NF_{+,\mathcal{B}}(x, y)),$$

written in pthe same form),

$$Alength : N \to N,$$

(the length of the Cantor normal-form of an ordinal)

$$\cdot : N \to N \to N,$$

written infix, (for multiplication of an ordinal by a natural number)

$$\Omega. : N \to N$$

(the enumeration of the infinite cardinals)

$$\cdot^{-} : N \to N,$$

(written a^{-} *for* $\cdot^{-}a$*, which we fuse only for cardinals, the most important definition is* $\Omega^{-}_{a+1} = \Omega_a$*).*

We define $a \preceq b \prec c := a \preceq b \wedge b \prec c$ *the same for similar situations, and* $\cdot^{\prec} := \lambda x.\lambda y.(y \prec x \wedge_{\mathcal{B}} y \;\eta\; OT)$, $\cdot^{\prec} : N \to \mathcal{P}^{dec}(N)$, *we write* $a^{\prec}$ *for* $\cdot^{\prec}a$.

We have $a^{\prec} \subset OT$ *and will usually omit the superscript* $\prec$.

Similar to the subsets of the natural numbers, we define abbreviations for the quantification over elements of OT $\forall_{\prec} := \lambda x, f.(\forall y \;\eta\; x^{\prec}.fx)$,

$\exists_{\prec} := \lambda x, f.(\exists y \;\eta\; x^{\prec}.fx)$,

$\forall_{\prec}, \exists_{\prec} : N \to (N \to U) \to U$, *and we write*

$\forall x \prec a.\phi$ *for* $\forall_{\prec} a(\lambda x.\phi)$,

$\forall_{typ} x \prec a.A := \forall x \in N.x \;\eta\; a^{\prec} \wedge A$,

similarly for $\exists$, *and have* $x : N \to A$ *type, then* $\forall_{typ} x \prec a.A, \exists x \prec_{typ} a.A,$ *type.*

$\forall x \preceq a.\phi := \forall x \;\eta\; (a + 1_{OT})^{\prec}.\phi$,

similarly for $\exists$.

We assume that

(a) $\forall x, y \;\eta\; OT.\forall z \in N.\Omega_x, \tau(x), x + y, x \cdot z \;\eta\; OT \wedge (y \;\eta\; \tau(x)^{-} \to x[y] \;\eta\; OT)$

(b) $\forall x, y, z \;\eta\; OT.\neg(x \prec x) \wedge (x \prec y \to y \prec z \to x \prec z) \wedge$
$(x \prec y \vee x = y \vee y \prec x)$,
$\forall x, y \;\eta\; OT.x \preceq y \leftrightarrow (x = y \vee x \prec y)$

(c) $\forall x, y, z \;\eta\; OT.x + (y + z) = (x + y) + z.$

(d) $\forall x \;\eta\; OT.x \;\not\eta\; A \leftrightarrow (x = 0_{OT} \vee \exists y, z \;\eta\; OT.z \neq 0_{OT} \wedge NF_{+}(y, z) \wedge x = y + z)$,
$\forall x, y \;\eta\; OT.NF_{+}(x, y) \to y \neq 0_{OT} \to (Alength(x) <_N Alength(x + y) \wedge Alength(y) <_N Alength(x + y))$,

$$\forall x \;\eta\; OT, y \;\eta\; OT. \quad NF_{+}(x, y) \vee x + y = y \vee \exists z, z' \;\eta\; OT.(x = z + z' \wedge z' \neq 0_{OT} \wedge NF_{+}(z, z') \wedge x + y = z + y).$$

(e) $\forall x \;\eta\; OT.x \;\eta\; Lim \vee (\exists y \;\eta\; OT.x = y + 1_{OT}) \vee x = 0_{OT}$
$\forall x \;\eta\; OT.x + 1_{OT} \;\not\eta\; Lim \wedge x + 1_{OT} \neq 0_{OT} \wedge 0_{OT} \;\not\eta\; Lim.$

(f) $\forall x \;\eta\; OT.\neg(x \prec 0_{OT})$,
$\forall x \;\eta\; OT.0_{OT} + x = x + 0_{OT} = x$
$\forall x \;\eta\; OT.x \prec y + 1_{OT} \leftrightarrow x \preceq y$,
$\forall x, y, z \;\eta\; OT.y \prec z \to x + y \prec x + z.$
$\forall x \;\eta\; OT.NF_{+}(x, 1_{OT})$

(g) $\forall x\ \eta\ OT.x \prec \omega \leftrightarrow \exists n \in N.x = 1_{OT} \cdot n$
$\forall x\ \eta\ OT.x \cdot 0 = 0_{OT},\ x \cdot (Sn) = (x \cdot n) + x.$

(h) $\Omega_{0_{OT}} = \omega,\ \Omega_I = I,$
$\forall x, y\ \eta\ OT.x \prec y \rightarrow \Omega_x \prec \Omega_y$

(i) $\forall x\ \eta\ OT.\Omega^-_{x+1_{OT}} = \Omega_x,$
$I^- = \omega^- = {1_{OT}}^- = {0_{OT}}^- = 0_{OT}.$

(j) $\forall x\ \eta\ OT.x\ \eta\ R \leftrightarrow x = I \vee \exists y\ \eta\ OT.x = \Omega_{y+1_{OT}},$
$\forall x, y\ \eta\ R.x \prec y \neq I \rightarrow x \preceq y^-$

(k) $\forall x\ \eta\ OT.\tau(x) \preceq x \wedge \tau(x)\ \eta\ R \cup \{0_{OT}, 1_{OT}, \omega\}$

(l) $\forall x\ \eta\ R \cup \{0_{OT}, 1_{OT}, \omega\}.\forall y\ \eta\ OT.(y \prec x \rightarrow x^- + y \prec x) \wedge NF_{\hat{+}}(x, x).$

(m) $\forall x, y, z\ \eta\ OT.z \prec y \rightarrow NF_+(x, y) \rightarrow NF_+(x, z).$

Further we assume the following laws (which are minor modifications to [Buc90]):

(F1) $\forall x\ \eta\ Lim.\forall y, z\ \eta\ \tau(x)^{\prec}.(y \preceq x[y] \prec x) \wedge (y \prec z \rightarrow x[y] \prec x[z])$

(F2) $\forall x\ \eta\ Lim.\forall y\ \eta\ OT.x[0_{OT}] \preceq y \prec x \rightarrow \exists z\ \eta\ \tau(x)^{\prec}.x[z] \preceq y \prec x[z + 1_{OT}]$

(F3) $\forall x\ \eta\ R \cup \{0_{OT}, 1_{OT}\}.\tau(x) = x \wedge \forall y\ \eta\ \tau(x)^{\prec}.x[y] = x^- + y$

(F4) $\forall x\ \eta\ Lim.\forall y\ \eta\ \tau(x)^{\prec}.\forall z\ \eta\ Lim.x[y] \prec z \preceq x[y + 1_{OT}] \rightarrow x[y] \preceq z[0_{OT}]$

(F5) $\forall x\ \eta\ Lim.\forall y\ \eta\ \tau(x)^{\prec} \cap Lim.\tau(x[y]) = \tau(y) \wedge$
$((\forall z\ \eta\ \tau(y)^{\prec}.x[y[z]] = x[y][z])$
$\vee(\tau(y) = \omega \wedge \forall z \prec \omega.x[y[z]] = x[y][z + 1_{OT}]))$

(F6) $\forall x\ \eta\ OT.\tau(x) \preceq x$

(F7) $\forall x\ \eta\ OT.(NF_+(x, y) \wedge y \neq 0_{OT}) \rightarrow \tau(x + y) = \tau(y) \wedge \forall z\ \eta\ \tau(y)^{\prec}.(x + y)[z] = x + (y[z])$

(F8) $\tau(1_{OT}) = 1_{OT} \wedge 1_{OT}[0_{OT}] = 0_{OT}.$

So (F1), (F2) express that the fundamental sequences are ascending and approximate the ordinal, (F3) tells, that for regular cardinals, the fundamental sequence is more or less the identity (we start with $\tau(a)^-$ to avoid fixed points of $\lambda x.\Omega_x$). (F4) is the Bachmann property, together with (F5) it guarantees, that the correct trees are highly uniform. These properties allow to prove lemma 8.15, especially 8.15 (a), which expresses, that for these uniform trees, there is some correspondence between the trees in $W(X)$ and the number of the branch, they belong to. (F6)-(F8) are usual properties of fundamental sequences. We will in the following, if there is no confusion, write $0, 1$ instead of $0_{OT}, 1_{OT}$.

Now we are ready to define $W(X)$:

Definition 8.11 *(a) We define the sets $\tau'(a)$, , which we will include in $\tau(a)^X$ in any case. $\tau'(a)$ will contain $\tau(a)^-$ except in the case $\tau(a) = 0$, to guarantee some information on $\tau(a)$, and in the case $\tau(a) = \omega$, $\tau'(a)$ will be the whole set $\tau(a)^{\prec}$, which makes sense, since $\omega^{\prec}$ is trivially well-ordered.*

$\tau' := \lambda x.\ \underline{if}\ (x =_{N,\mathcal{B}} 0 \vee x \not\eta_{\mathcal{B}}\ OT)\ \underline{then}\ \emptyset_{\mathcal{P}}\ \underline{else}\ \underline{if}\ \tau(x) =_{N,\mathcal{B}} \omega\ \underline{then}\ \omega^{\prec}\ \underline{else}\ \{\tau(x)^{-}\}$,
that is, in a form, that can be read more easily,

$$\tau'(a) = \begin{cases} \emptyset_{\mathcal{P}} & \text{if } a =_{N,\mathcal{B}} 0 \vee a \not\eta_{\mathcal{B}}\ OT \\ \omega^{\prec} & \text{if } \tau(x) = \omega \\ \{\tau(a)^{-}\} & \text{otherwise.} \end{cases}$$

$\tau' : N \to \mathcal{P}(N)$ *(note that we could define* $\tau' : N \to \mathcal{P}^{\mathrm{dec}}(N)$*, but do not need this).*

(b) *We first give the definition of $W(X)$ for X a subclass. The possibility to define $W(X)$ for classes (in $\Delta^1_2 - CA + BI$ this can not be generally be done) is the reason, why $ML^i_1W_R$ has strength $\psi_{\Omega_1}(\Omega_{I+\omega})$, rather than $\psi_{\Omega_1}(\Omega_I)$): After having constructed ausgezeichnete Mengen, which exhaust the ordinals up to I, we can form "ausgezeichnete Klassen", distinguished classes, which exhaust the ordinals up to Ω_{I+n} for every $n : N$, by building the union of all ausgezeichnete Mengen and iterating the step to $W(X)$ finitely many times.*

x, y, z are variables

$$\tau^{A,z}_{Cl,y}(x) := (y\ \eta_{Cl,z}\ A \wedge y\ \eta\ OT \wedge y \prec \tau(x)) \vee y\ \eta\ \tau'(x)$$

that is, if $z = y$

$$\tau^{A,y}_{Cl,y}(x) = (A \cap_{Cl,y} \tau(x)^{\prec}) \cup_{Cl,y} \tau'(x)$$

We will usually not mention the variable z, sometimes not the variable y.
We have, if $\Gamma \Rightarrow A \in \mathcal{C}l_z(N)$ for some variable z, and $x \neq y$ is a new variable, then

$$\Gamma, x : N \Rightarrow \tau^{A,z}_{Cl,y} \in \mathcal{C}l_y(N)$$

and

$$\Gamma, x : N \Rightarrow p : \tau^{A,z}_{Cl,y}(x) \subset_{Cl,y} OT \text{ for some } p$$

$$\Gamma \Rightarrow \tau^{A,z}_{Cl,y}(0) \cong \emptyset,$$

therefore

$$\Gamma \Rightarrow \forall y \in N. y \not\eta\ \tau^{A,z}_{Cl,y}(0),$$

$$B_{1,Cl,y,x,z}(A) := \Sigma y \in N. y\ \eta\ \tau^{A,z}_{Cl,y}(x)$$

(if x, y are variables)

$$W_{1,Cl,y,z}(A) := Wx \in N. B_{1,Cl,y,x,z}(A),$$

where x is a new variable, y is a new variable.
and we define

$$\begin{aligned} Cor_{Cl,y,z}(A)(\alpha) := \quad & \forall x \in N. \forall u \in B_{1,Cl,y,x,z}(A) \to W_{1,Cl,y,z}(A). \\ & sup(x,u) \preceq_{N,B_{1,Cl,y,x,z}(A)} \alpha \to \forall v \in B_{1,Cl,y,x,z}(A). label(uv) = x[v0] \end{aligned}$$

where u, v, x are new variables, and

$$label := \lambda \alpha. R(\alpha, (u, v, w)u),$$

$label(sup(r,s)) = r$, $label : W_{1,Cl,y}(A) \to N$.
and have, if $\Gamma \Rightarrow A \in \mathcal{C}l_z(N)$ and α is a new variable, then

$$\Gamma, \alpha : W_{1,Cl,y,z}(A) \Rightarrow Cor_{Cl,y,z}(N)(\alpha) \text{ type.}$$

$$W_{Cl,y,z}(A) := \exists\alpha \in W_{1,Cl,y,z}(A).Cor_{Cl,y,z}(\alpha) \wedge label(\alpha) = y \wedge label(\alpha)\ \eta\ OT$$

(where α is a new variable).
Therefore, if we have $sup(r,s) : W_{1,Cl,y,z}(A)$ such that $Cor_{Cl,y,z}(sup(r,s))$, then the predecessors of $sup(r,s)$ form a tree, verifying, then r is in the least set X with $\forall\gamma\ \eta\ OT.\tau^{A}_{Cl,y,z}(\gamma) \subset X \to \gamma\ \eta\ X$.
For $X, Y \in \mathcal{C}l_z(N)$, z', z'' are new variable, we define
$\mathcal{A}^{X}_{Cl,y,z}(Y) := y\ \eta\ OT \wedge \forall z'\ \eta\ \tau^{X,z}_{Cl,z''}(y).y[z']\ \eta\ Y$, *and have*
$\mathcal{A}^{X}_{Cl,y,z}(Y) \in \mathcal{C}l_y(N)$. *We will usually omit the Indices Cl and indices and superscripts for variables, assuming that for the z we choose the variable for which we habe proven $\Gamma \Rightarrow A \in \mathcal{C}l_z(N)$, and that y is the variable usually taking for classes, and choosing the bounded variables, such that they do not cause any problems.*

(c) *Now we define $W(X)$ for $X : \mathcal{P}(N)$.*

$\tau := \lambda X, x.((X \cap \tau(x)^{\prec}) \cup \tau(x)')$. $\tau : \mathcal{P}(N) \to N \to \mathcal{P}(N)$. *and* $\forall X \in \mathcal{P}(N).\forall x \in N.\tau X x \subset OT$
We write $\tau(a)^X$ for $\tau X a$.

We define $B_1 := \lambda X.\lambda x.\Sigma z \in N.z\ \eta\ \tau(x)^X$ (written $B_1(A)$ for $B_1 A$)
$B_1 : \mathcal{P}(N) \to N \to \mathcal{P}(N)$
$W_1 := \lambda X.W x \in N.B_1(X)x$
$W_1 : \mathcal{P}(N) \to U$

indexCor $Cor_{:} = \lambda X, \alpha.\forall r \in N.\forall s \in (B_1(A)r \to W_1(A))$.
$\quad sup(r,s) \preceq \alpha \to \forall z \in B_{1,Cl}(A)r.label(sz) = r[z0]$,

where $label := \lambda X.\lambda\alpha.R(\alpha, (u,v,w)u)$,
$label(sup(r,s)) = r$, $label : \Pi X \in \mathcal{P}(N).W_1(A) \to N$.
$Cor : \Pi X \in \mathcal{P}(N).W_1 X \to U$,

$W := \lambda X.\lambda y.\exists\alpha \in W_1 X.Cor X\alpha \wedge (label\alpha) = y \wedge y\ \eta\ OT$
$W : \mathcal{P}(N) \to N \to \mathcal{P}(N)$,
and we have $\forall X \in \mathcal{P}(N), y \in OT.WXy \subset OT$.
We write $W_1(X)$ for $W_1 X$, $W(X)$ for WX.

We define $\mathcal{A} := \lambda X, Y, y.y\ \eta\ OT \wedge \forall z\ \eta\ \tau(y)^X.y[z]\ \eta\ Y$, $\mathcal{A} : \mathcal{P}(N) \to \mathcal{P}(N) \to \mathcal{P}(N)$.
We write $\mathcal{A}^X(Y)$ for $\mathcal{A}XY$.

Lemma 8.12 *Now we have induction over $W(X)$, which we prove by using the recursion over $W_1(X)$:*

(a) *If* $\Gamma \Rightarrow X, Y \in \mathcal{C}l_y(N)$, *then*
$\Gamma \Rightarrow W(X) \in \mathcal{C}l_y(N)$,
$\Gamma \Rightarrow \mathcal{A}^X(Y) \in \mathcal{C}l_y(N)$,
$\Gamma \Rightarrow W(X), \mathcal{A}^X(Y) \subset OT$.
$\forall X, Y \in \mathcal{P}(N).W(X) \subset OT \wedge \mathcal{A}^X(Y) \subset OT$.

(b) *For* $X \in \mathcal{C}l_y(N)$ *or* $X : \mathcal{P}(N)$ *follows* $\mathcal{A}^X(W(X)) \subset W(X)$

(c) *If* $X, Y \in \mathcal{C}l_y(N)$ *or* $X, Y : \mathcal{P}(N)$, *then*
if $\Gamma \Rightarrow \mathcal{A}^X(Z) \cap W(X) \subset Z$, *then* $\Gamma \Rightarrow W(X) \subset Z$.
More precisely we have
If $\Gamma \Rightarrow X \in \mathcal{C}l_y(N)$, $\Gamma \Rightarrow Z \in \mathcal{C}l_z(N)$*and* $\Gamma \Rightarrow \mathcal{A}^X(Z) \cap W(X) \subset Z$, *then* $\Gamma \Rightarrow W(X) \subset Z$.

If $\forall X, Z \in \mathcal{P}(N).\mathcal{A}^X(Z) \cap W(X) \subset Z \to W(X) \subset Z$,
If $\Gamma \Rightarrow Z \in \mathcal{C}l_y(N)$, *then* $\Gamma \Rightarrow \forall X \in \mathcal{P}(N).\mathcal{A}^X(Z) \subset Z \to W(X) \subset Z$. *If* $\Gamma \Rightarrow X \in \mathcal{C}l_y(N)$, *then* $\Gamma \Rightarrow \forall Z \in \mathcal{P}(N).\mathcal{A}^X(Z) \subset Z \to W(X) \subset Z$.

Proof: (a) is obvious. (Note, that we really have, if $\Gamma \Rightarrow X \in \mathcal{C}l_y(N)$, then $\Gamma \Rightarrow W(X) \in \mathcal{C}l_y(N)$ with the same y.
(b) statement for classes:
Let a be a fresh variables. Assume $a\ \eta\ \mathcal{A}^X(W(X))$, that is

$$a\ \eta\ OT \wedge \forall x\ \eta\ \tau(a)^X.a[x]\ \eta\ W(X)$$

Therefore

$$\forall x \in (B_{1,Cl}(X)[x/a]).\exists \beta \in W_1(X).Cor(\beta) \wedge label(\beta) = a[x0]$$

Let $\alpha := sup(a, s)$ where for $x : (B_{1,Cl}(X)[x/a])$ we have $sx =_{W_1(X)} \beta$ such that $Cor(\beta) \wedge label(\beta) =_{W_1(X)} a[x0]$. Since we have $Cor(sx)$ for $x : (B_1(X)[x/a])$,
$\forall x \in (B_1(X)[x/a]).label(sx) = a[x0]$ and $\gamma \preceq \alpha \leftrightarrow \gamma =_{W_1(X)} \alpha \vee \exists x \in (B_1(X)[x/a]).\gamma \preceq sx$ follows $Cor(\alpha)$.
The statement for subsets follows similarly.
(c) statement for classes:
Assume $X, Z \in \mathcal{C}l_y(N)$, $\mathcal{A}^X(Z) \cap W(X) \subset Z$.
We show $\forall \alpha \in W_1(X).Cor(\alpha) \to label(\alpha)\ \eta\ Z$ by Induction on $W_1(X)$.
Let $\alpha =_{W_1(X)} sup(r, s)$, $Cor(\alpha)$.Then $r\ \eta\ OT$, $\forall z \in B_1(X)[x/r].Cor(sz)$. By IH we have, for $z : N$, $p : z\ \eta\ \tau(a)^X$, since $< z, p >\eta\ B_1(X)[x/r]$, $label(\alpha)[z0] = s < z, p >\eta\ Z$, therefore $r = label(\alpha)\ \eta\ \mathcal{A}^X(Z) \cap W(X) \subset Z$.
The statements for subsets follows similarly.

Lemma 8.13 *Assume* $X \in \mathcal{C}l_y(N)$ *or* $X : \mathcal{P}(N)$. *Then* $\forall x, y\ \eta\ W(X).x + y\ \eta\ W(X)$.

Proof:
Statement for $X : \mathcal{P}(N)$.
We first prove $(NF_+(a, b) \wedge a, b\ \eta\ W(X)) \to a + b\ \eta\ W(X)$
Let $a\ \eta\ W(X)$, and $Y := \lambda y.NF_+(a, y) \to a + y\ \eta\ W(X)$, and have $Y : \mathcal{P}(N)$.
We show $\mathcal{A}^X(Y) \subset Y$ (then follows $W(X) \subset Y$).
Assume $b\ \eta\ \mathcal{A}^X(Y)$, $NF_+(a, b)$.
If $b = 0$, then $a + b\ \eta\ W(X)$, $b\ \eta\ Y$. Otherwise $\tau(a + b) = \tau(b)$ and $\forall z\ \eta\ \tau(b).(a + b)[z] = a + (b[z])$, and $\forall z\ \eta\ \tau(b)^X.b[z]\ \eta\ Y$, e.g. since $NF_+(a, b[z])$, $\forall z\ \eta\ \tau(b)^X \cong \tau(a + b)^X.(a + b)[z] = a + (b[z])\ \eta\ W(X)$, $a + b\ \eta\ W(X)$.

Now we prove $a + b\ \eta\ W(X) \to NF_+(a, b) \to a\ \eta\ W(X)$.
Let $Y := \lambda y.\forall x, z\ \eta\ OT.NF_+(x, z) \wedge x + z = y \to x\ \eta\ W(X)$.
Let $c\ \eta\ \mathcal{A}^X(Y) \cap W(X)$, $c = a + b$, $NF_+(a, b)$. If $b = 0$, then $a = c\ \eta\ W(X)$, and if $b \neq 0$, then $\tau(c) = \tau(b)$ and $\forall \xi\ \eta\ \tau(b)^{\prec}.c[\xi] = a + b[\xi]$. By (F2), (F7) and (F8) follows $\tau(b) \neq 0$, therefore $a + b[0]\ \eta\ Y$, $NF_+(a, b[0])$, $a\ \eta\ W(X)$.

Now follows the assertion, by Induction on $Alength(a)$: If $NF_+(a, b)$ follows $a+b\ \eta\ W(X)$, if $a + b = b$ the assertion is trivial and if $a = c + d$ such that $NF_+(c, d)$, $a + b = c + b$ and $Alength(c) <_N Alength(a)$ and $c\ \eta\ W(X)$, and the assertion follows by IH.

Lemma 8.14 *We state some easily proven results on* $W(X)$.
Assume $X, X' \in \mathcal{C}l_y(N)$ *or* $X, X' : \mathcal{P}(N)$, $a, b : N$.

(a) $0\ \eta\ W(X) \wedge \forall x\ \eta\ W(X).x + 1\ \eta\ W(X) \wedge \forall x \preceq \omega.x\ \eta\ W(X)$.

(b) $(X \cap a \cong X' \cap a \wedge (\forall x \prec b.\tau(x) \preceq a)) \to W(X) \cap b \cong W(X') \cap b.$

(c) If $X \subset W(X)$*, then* $\forall x \,\eta\, R.x^- \,\eta\, W(X) \to x \,\eta\, W(X)$*.*

(d) $X \subset W(X) \to \{\omega, \Omega_1, I\} \subset W(X)$*.*

Proof: (a) $0 \,\eta\, \mathcal{A}^X(W(X)) \subset W(X)$ and, if $a \,\eta\, W(X)$, then $a+1 \,\eta\, \mathcal{A}^X(W(X)) \subset W(X)$, therefore by induction on N

$$\forall n \in N.1 \cdot n \,\eta\, W(X)$$

and

$$\forall x \prec \omega.x \,\eta\, W(X)$$

It follows $\omega \,\eta\, \mathcal{A}^X(W(X)) \subset W(X)$, therefore $\forall x \preceq \omega.x \,\eta\, W(X)$.
(b) Let $Y := \lambda y.y \prec b \to y \,\eta\, W(X')$, in the case of $X, X' : \mathcal{P}(N)$, $Y := y \prec b \to y \,\eta\, W(X')$ in the case of classes. We proof $\mathcal{A}^X(Y) \subset Y$. This implies $W(X) \cap b \subset W(X')$. Assume $\forall x \,\eta\, \tau(y)^X.y[x] \,\eta\, Y$, $y \prec b$. Then $\tau(y)^X \cong \tau(y)^{X'}$, $\forall x \,\eta\, \tau(y)^X.y[x] \,\eta\, W(X')$, therefore $y \,\eta\, \mathcal{A}^{X'}(W(X'))$ and thus $y \,\eta\, W(X')$.
(c) If $x^- \,\eta\, W(X)$, then $x[\tau(x)^-] = x^- + x^- \,\eta\, W(X)$ (using lemma 8.13) and since $\tau(x)^X \subset X \cup \{x^-\} \subset W(X)$ follows by 8.13 $\forall \xi \,\eta\, \tau(x) \cap X.x[\xi] = x^- + \xi \,\eta\, W(X)$.
(d) $\omega \,\eta\, W(X)$, therefore follows the assertion by $\Omega_1^- = \omega$, $I^- = 0$.

Lemma 8.15 *Let* $X : \mathcal{P}(N)$ *or* $X \in \mathcal{C}l_y(N)$*,* $a, b \,\eta\, OT$*.*

(a) $a \,\eta\, Lim \wedge x \prec \tau(a) \wedge a[x] \preceq b \preceq a[x+1] \wedge b \,\eta\, W(X)) \to x \,\eta\, W(X)$*, esp.*
$\forall a \,\eta\, Lim, x \,\eta\, \tau(a)^{\prec}.a[x] \,\eta\, W(X) \to x \,\eta\, W(X)$*.*

(b) $a \,\eta\, Lim \wedge a[0] \preceq b \prec a \wedge b \,\eta\, W(X)) \to a[0] \,\eta\, W(X)$*.*

(c) $a \,\eta\, Lim \to b \,\eta\, W(X) \cap \tau(a)^{\prec} \to \exists x \,\eta\, (W(X) \cap \tau(a)^{\prec}).b \preceq a[x]$*.*

(d) $X \subset W(X) \to \forall x \,\eta\, W(X).\tau(x) \,\eta\, W(X)$*.*

Proof: (a) Let $Y := \forall x \,\eta\, \tau(a)^{\prec}.a[x] \preceq y \prec a[x+1] \to y \,\eta\, W(X) \to x \,\eta\, W(X)$, $Y \in \mathcal{C}l_y(N)$. We show $\mathcal{A}^X(Y) \subset Y$, therefore by lemma 8.12 (c) $W(X) \subset Y$
Let $b \,\eta\, \mathcal{A}^X(Y)$.
Case 1: $a[x] \prec b \preceq a[x+1]$: Then $a[x] \preceq b[0] \preceq b[\tau(b)^-] \prec a[x+1]$ and by $b[\tau(b)^-] \,\eta\, Y$, $x \,\eta\, X$.
Case 2: $\neg(a[x] \prec b)$. Then $a[x] = b$.
Subcase $x = 0$: $x \,\eta\, W(X)$
Subcase $x = x' + 1$: Then $\tau(a)^- \preceq x'$, $a[x'] \prec b \preceq a[x'+1]$ and by Case 1 $x' \,\eta\, W(X)$ and by 8.14 (a) follows $x \,\eta\, W(X)$.
Subcase $x \,\eta\, Lim$: Then
$\tau(b) = \tau(x) \wedge ((\forall y \prec \tau(x).a[x[y]] = b[y]) \vee (\tau(b) = \omega \wedge \forall y \prec \omega.a[x[y]] = b[y+1]))$.
Therefore $\tau(b)^X = \tau(x)^X$ and $\forall y \,\eta\, \tau(b)^X.b[y] \,\eta\, W(X)$. By $b \,\eta\, \mathcal{A}^X(Y)$, since $\tau(b) = \tau(x) = \omega \to \forall y \prec \omega.y+1 \,\eta\, \tau(b)^X$ follows $\forall y \,\eta\, \tau(x)^X.x[y] \,\eta\, W(X)$, therefore $x \,\eta\, W(X)$.
(b) Let $Y := a[0] \preceq y \prec a \to a[0] \,\eta\, W(X)$, $Y \in \mathcal{C}l_y(N)$. We show $\mathcal{A}^X(Y) \cap W(X) \subset Y$, therefore by lemma 8.12 (c) $W(X) \subset Y$.
Let $b \,\eta\, \mathcal{A}^X(Y) \cap W(X)$.
If $b = a[0]$ follows $a[0] \,\eta\, W(X)$ and if $a[0] \prec b$ follows $a[x] \prec b \preceq a[x+1]$ for some $x \,\eta\, \tau(a)^{\prec}$, $a[0] \preceq a[x] \preceq b[0] \preceq b[\tau(b)^-] \,\eta\, Y$, $a[0] \,\eta\, W(X)$

(c) If $b \preceq a[0]$ we chose $x := 0 \; \eta \; W(X)$ If $a[0] \prec b$ we have by (F2) $a[x] \preceq b \prec a[x+1]$ for some $x \; \eta \; \tau(a)^{\prec}$. By (a) we obtain $x \; \eta \; (W(X) \cap \tau(a)^{\prec})$.
(d) If $a \; \eta \; W(X)$, then $a[\tau(a)^{-}] \; \eta \; W(X)$, $\tau(a)^{-} \; \eta \; W(X)$ by (a), $\tau(a) \; \eta \; W(X)$ by 8.14 (a), (c) and (d).

Definition 8.16 **Now we define** *the meaning of "X is ausgezeichnete Menge":*
Let $X, Y : \mathcal{P}(N)$ *or* $X, Y \in \mathcal{C}l_y(N)$.
$X|a := X \cap (a+1)$.
$X \sqsubseteq Y := \forall x \; \eta \; X.X|a \cong Y|a$ *(this is equivalent to* $X \subset Y \wedge \forall x \; \eta \; X.Y \cap x \subset X$, *X is a segment of Y).*
$Ag(X) := X \subset OT \wedge X \sqsubseteq W(X)$, *X is an "ausgezeichnete Menge", a "distinguished set".*
$Prog(X, Y) := \forall x \; \eta \; X.X \cap x \subset Y \rightarrow x \; \eta \; Y$.
$Prog(Y) := \forall x \prec \Omega_1.OT \cap x \subset Y \rightarrow x \; \eta \; Y$.

Lemma 8.17 *From the induction over* $W(X)$ *we conclude induction over ausgezeichnete Mengen:*
Assume $X, Y : \mathcal{P}(N)$ *or* $X, Y \in \mathcal{C}l_y(N)$, *such that* $Ag(X)$.

(a) $X \sqsubseteq \mathcal{A}^X(X)$.

(b) $X \cap \mathcal{A}^X(Y) \subset Y \rightarrow X \subset Y$.

(c) $Prog(X, Y) \rightarrow X \subset Y$.

(d) $X \cap \Omega_1 \sqsubseteq OT$.

(e) $Prog(Y) \rightarrow X \cap \Omega_1 \subset Y$.

Proof: We treat here the case $X, Y \; \eta \; \mathcal{C}l_y(N)$.
(a) $a \; \eta \; X \rightarrow X|a \cong W(X)|a \rightarrow \mathcal{A}^X(X)|a \cong \mathcal{A}^X(W(X))|a \cong W(X)|a \cong X|a$.
(b) Let $Y' := y \; \eta \; X \rightarrow y \; \eta \; Y \in \mathcal{C}l_y(N)$ and assume $X \cap \mathcal{A}^X(Y) \subset Y$. Then, since $X \subset \mathcal{A}^X(X)$, $\forall y \; \eta \; (X \cap \mathcal{A}^X(Y')).\forall z \; \eta \; \tau(y)^X.y[z] \; \eta \; Y$ therefore $X \cap \mathcal{A}^X(Y') \subset X \cap \mathcal{A}^X(Y) \subset Y$, therefore $\mathcal{A}^X(Y') \subset Y'$, $X \subset W(X) \subset Y'$, $X \subset Y$.
(c) Let $Y' := X \cap y \subset Y$, we have $\Gamma \Rightarrow Y' \in \mathcal{C}l_y(N)$ (more precisely $Y' := \forall u \in N.u \; \eta \; X \wedge u \prec y \wedge u \; \eta \; OT \rightarrow u \; \eta \; Y$, where u is a new variable).
We show

(∗)
$$X \cap_{Cl} \mathcal{A}^X_{Cl}(Y') \subset_{Cl} Y'$$

Proof: Let $a \; \eta \; X \cap \mathcal{A}^X(Y')$ and assume $b \; \eta \; X \cap a$. From $a \; \eta \; X \sqsubseteq W(X)$ and $b \; \eta \; X \cap a$ follows by lemma 8.15 (a) $b \preceq a[x]$ for some $x \; \eta \; W(X) \cap \tau(a)^{\prec} \cong X \cap \tau(a)^{\prec}$, (since $\tau(a) \preceq a \; \eta \; X \sqsubseteq W(X)$. By $a \; \eta \; \mathcal{A}^X(Y')$ follows $a[x] \; \eta \; Y'$, $X \cap b \subset X \cap a[x] \subset Y$, and by $Prog(X, Y)$ and $b \; \eta \; X$ follows $b \; \eta \; Y$ for arbitrary $b \; \eta \; X \cap a$, therefore $X \cap a \subset Y$, $a \; \eta \; Y'$, and we have (∗).
By (b) and (∗) follows $X \subset Y'$, that is $\forall x \; \eta \; X.X \cap x \subset Y$ and using again $Prog(X, Y)$, $\forall x \; \eta \; X.x \; \eta \; Y$.
(d) Let $Y := y \prec \Omega_1 \rightarrow \forall z \prec y.z \; \eta \; X$ (z a new variable). We prove $X \cap \mathcal{A}^X(Y) \subset Y$. By (b) follows the assertion.
Assume $a \; \eta \; X \cap \Omega_1$, $a \; \eta \; \mathcal{A}^X(Y)$. From $\forall x \; \eta \; \tau(a)^X.a[x] \; \eta \; Y$ we conclude $a[x] \subset X$, $a[x] \; \eta \; W(X)$, and by $a[x] \prec a \; \eta \; X \sqsubseteq W(X)$, $a[x] \; \eta \; X$, therefore $\forall x \; \eta \; \tau(a)^X.OT|a[x] \subset X$. Since $\tau(a) \; \eta \; \{0, 1, \omega\}$, follows $\tau(a)^X \cong \tau(a)$, $a \subset X$.
(e) Assuming $Prog(Y)$ follows by (d) $\forall x \; \eta \; X \cap \Omega_1.X \cap x \subset Y \rightarrow x \; \eta \; Y$, therefore with $Y' := y \prec \Omega_1 \rightarrow y \; \eta \; Y$ $Prog(X, Y')$ and by (c) $X \cap \Omega_1 \subset Y$.

Lemma 8.18 *Let $\Gamma \Rightarrow X \in \mathcal{C}l_y(N)$ or $X : \mathcal{P}(N)$, $Ag(X)$. Then $\forall x \;\eta\; X.\tau(x) \;\eta\; X$.*

Proof:
If $a \;\eta\; X$, then $a \;\eta\; W(X)$, $\tau(a) \;\eta\; W(X)$ by 8.15 (d), and, since $\tau(a) \preceq a \;\eta\; X \sqsubseteq W(X)$, $\tau(a) \;\eta\; X$.

Lemma 8.19 *We state, that ausgezeichnete Mengen are unique, in the following sense: Let $X_i \in \mathcal{C}l_y(N)$ or $X_i : \mathcal{P}(N)$. If $X_i \cong W(X_i) \cap a$ $(i = 1, 2)$, then $X_0 \cong X_1$.*

Proof: We treat the case $X_i \in \mathcal{C}l_y(N)$.
We have

(1) If $U \in \mathcal{C}l_y(N)$ and $Prog(X_0 \cup X_1, U)$ then $X_0 \cup X_1 \subset U$

Proof: Assume $Prog(X_0 \cup X_1, U)$. Assume b such that $X_0 \cap b \subset U$. Then $\forall c \prec b.(c \;\eta\; X_1 \wedge X_1 \cap c \subset U) \to c \;\eta\; U$. Therefore we have $Prog(X_1 \cap b, U)$ and by lemma 8.17 (c), since we have $Ag(X_1)$ $X_1 \cap b \subset U$. By $Prog(X_0 \cup X_1, U)$ we have $b \;\eta\; U$ for b with $X_0 \cap b \subset U$. Therefore we have $Prog(X_0, U)$ and using $Ag(X_0)$ and lemma 8.17 (c) $X_0 \subset U$. Similarly we conclude $X_1 \subset U$. and we have (1).

We show

(2) $$Prog(X_0 \cup X_1, X_0 \cap X_1)$$

Proof: Assume $b \;\eta\; X_0 \cup X_1$ and $(X_0 \cup X_1) \cap b \subset X_0 \cap X_1$. Then we have $b \prec a$ and $X_0 \cap b \cong X_1 \cap b$ and therefore $X_0|b \cong W(X_0)|b \cong W(X_1)|b \cong X_1|b$, and $b \;\eta\; X_0 \cap X_1$.

By (1) and (2) follows the assertion.

Next step is to define $\mathcal{W}$ as the union of all ausgezeichnete Mengen. It is itself a ausgezeichnete Klasse, closed under $\lambda x.\Omega_x$ for $x \prec I$, and will exhaust the ordinals up to I:

Definition 8.20 $\mathcal{W} := \exists X \in \mathcal{P}(N).Ag(X) \wedge y \;\eta\; X$, $\mathcal{W} \in \mathcal{C}l_y(N)$.

Lemma 8.21 *$\forall X \in \mathcal{P}(N).Ag(X) \leftrightarrow X \sqsubseteq \mathcal{W}$, that is: the ausgezeichnete Mengen are just segments of $\mathcal{W}$.*

Proof: "$\Rightarrow$": $X \subset \mathcal{W}$ is clear. Assume $a \;\eta\; X$, $b \;\eta\; \mathcal{W} \cap a$. Then there exists $Y \in \mathcal{P}(N)$ with $b \;\eta\; Y$ and $Ag(Y)$. $X' := X|b$, $Y' := Y|b$.
Then

$$\begin{aligned} W(X')|b &\cong W(X)|b && \text{by lemma 8.14 (b)} \\ &\cong X|b && X \sqsubseteq W(X),\ a \;\eta\; X,\ b \prec a \\ &\cong X' \end{aligned}$$

$$\begin{aligned} W(Y')|b &\cong W(Y)|b && \text{by lemma 8.14 (b)} \\ &\cong Y|b && Y \sqsubseteq W(Y),\ b \;\eta\; Y \\ &\cong Y' \end{aligned}$$

Therefore by lemma 8.19 $X' \cong Y'$, $b \;\eta\; Y' \cong X' \subset X$.

"$\Leftarrow$"If $a \;\eta\; X$, then there exists $Y : \mathcal{P}(N)$ such that $a \;\eta\; Y$ and $Ag(Y)$. By the proof of "$\Rightarrow$" follows $Y \sqsubseteq \mathcal{W}$, therefore $X|a \cong \mathcal{W}|a \cong Y|a \cong W(Y)|a$, therefore $W(X)|a \cong W(Y)|a \cong X|a$, and we have $Ag(X)$.

Lemma 8.22

$$Ag(\mathcal{W})$$

Proof: Let $a \ \eta \ \mathcal{W}$. Then $a \ \eta \ X \sqsubseteq \mathcal{W}$ for some $X : \mathcal{P}(N)$. It follows $\mathcal{W}|a \cong X|a$, therefore $W(\mathcal{W})|a \cong W(X)|a \cong X|a \cong \mathcal{W}|a$.

Lemma 8.23 *$\forall X \in \mathcal{P}(N).\forall x \ \eta \ OT.(X \cap x \cong W(X) \cap x \wedge x \ \eta \ W(X)) \to x \ \eta \ \mathcal{W}$.*

Proof: Assume $X : \mathcal{P}(N)$, $a \ \eta \ OT$, $X \cap a \cong W(X) \cap a \wedge a \ \eta \ W(X)$, and let $Y := W(X)|a$. Then $X \cap a \cong Y \cap a$ and therefore $Y \cong W(X)|a \cong W(Y)|a$, therefore $Ag(Y)$.

Lemma 8.24 *(a) $(\forall x \preceq \omega.x \ \eta \ \mathcal{W}) \wedge \forall x \ \eta \ \mathcal{W}.x + 1 \ \eta \ \mathcal{W}$.*

(b) $\forall x \ \eta \ OT.\Omega_x \ \eta \ \mathcal{W} \to \Omega_{x+1} \ \eta \ \mathcal{W}$.

Proof:
(a) $X := \lambda y.y \preceq \omega : \mathcal{P}(N)$, and we have $Ag(X)$.
further, if $X : \mathcal{P}(N)$, $Ag(X)$, $a \ \eta \ X$, then $X \cap (a+1) \cong W(X) \cap (a+1)$ and $a+1 \ \eta \ W(X)$, by 8.23 $a + 1 \ \eta \ \mathcal{W}$.
(b): Let $\Omega_s \ \eta \ X$, $Ag(X)$ for some $X : \mathcal{P}(N)$, $Y := W(X)$, $Y : \mathcal{P}(N)$, $a := \Omega_{s+1}$. Then $\Omega_{s+1} \ \eta \ Y$. $X \cap \Omega_s \cong Y \cap \Omega_s$, therefore $Y \cap a \cong W(X) \cap a \cong W(Y) \cap a$, by 8.23 follows $\Omega_{s+1} \ \eta \ \mathcal{W}$.

Lemma 8.25 *$\forall x \ \eta \ \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I.\tau(x) \ \eta \ \mathcal{W}$.*

Proof:
If $0 \neq a \ \eta \ \mathcal{A}^{\mathcal{W}}(\mathcal{W})$, then $a[\tau(a)^-] \ \eta \ \mathcal{W}$, therefore $a[\tau(a)^-] \ \eta \ X$ for some $X : \mathcal{P}(N)$ with $Ag(X)$. Therefore by lemma 8.15 $\tau(a)^- \ \eta \ W(X)$ and, since $\tau(a)^- \preceq a[\tau(a)^-] \ \eta \ X \sqsubseteq W(X)$, $\tau(a)^- \ \eta \ X$. We have $\tau(a)^- \ \eta \ (R \setminus \{I\}) \cup \{0, 1, \omega\}$, so by lemma 8.24 (b) (or trivially in the cases $\tau(a) \ \eta \ \{0, 1, \omega\}$) follows $\tau(a) \ \eta \ \mathcal{W}$.

Lemma 8.26 *$\forall x \ \eta \ \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I.(\exists X \in \mathcal{P}(N).X \cong \mathcal{W} \cap x) \to x \ \eta \ \mathcal{W}$.*

Proof:
Assume $a \ \eta \ \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I$.
By lemma 8.25 $\tau(a)\eta\mathcal{W}$.
Let $\tau(a) \prec a$ and $X \cong \mathcal{W} \cap a$. Then $\tau(a)\eta X$ and $Ag(X)$. We prove:

$$(*) \qquad W(X) \cap a \subset X$$

Let $b\eta W(X) \cap a$. Then there exists by lemma 8.15 $x\eta W(X) \cap \tau(a)^{\prec}$ such that $b \preceq a[x]$. Since $\tau(a)\eta X \sqsubseteq W(X)$, we conclude $x\eta\mathcal{W} \cap \tau(a)^{\prec}$ and $a[x]\eta\mathcal{W} \cap a \cong X$. Since $b\eta W(X)|a[x]$ and $a[x]\eta X$ follows $b\eta X$, and we have $(*)$.
Now $a\eta\mathcal{A}^{\mathcal{W}}(\mathcal{W})$ and $X \cong \mathcal{W} \cap a$, therefore $a\eta\mathcal{A}^X(X) \subset \mathcal{A}^X(W(X)) \cong W(X)$.
Since $X \cong W(X) \cap a$ and $a\eta W(X)$ follows $a\eta\mathcal{W}$ by lemma 8.23.

Lemma 8.27 *$W(\mathcal{W}) \cap I \cong \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I \cong \mathcal{W} \cap I$.*

Proof: We prove $\mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I \cong \mathcal{W} \cap I$:
"$\supset$": $Ag(\mathcal{W})$, therefore $\mathcal{W} \sqsubseteq \mathcal{A}^{\mathcal{W}}(\mathcal{W})$.
"$\subset$": Let $a \ \eta \ \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I$. Then $\tau(a) \ \eta \ \mathcal{W} \wedge \forall x \ \eta \ \mathcal{W} \cap \tau(a)^{\prec}.a[x] \ \eta \ \mathcal{W}$. Therefore exists $Q : \mathcal{P}(N)$ such that $Ag(Q) \wedge \tau(a) \ \eta \ Q$ and with $B := \Sigma x \in N.x \ \eta \ Q \cap a^{\prec}$ exists $s : B \to \mathcal{P}(N)$ such that $\forall x \in B.Ag(sx) \wedge a[x0] \ \eta \ sx$.
Let $P := \lambda y.y \ \eta \ Q \vee \exists x \in B.y \ \eta \ (sx)$, $P : \mathcal{P}(N)$. Then, since P is union of distinguished sets, follows $Ag(P)$, $P \sqsubseteq \mathcal{W}$.
Let $c \prec a$, $c \ \eta \ \mathcal{W}$, then there exists $x \ \eta \ W(\mathcal{W}) \cap \tau(a)^{\prec}$ ($\cong \mathcal{W} \cap \tau(a)^{\prec} \cong Q \cap \tau(a)^{\prec}$) such that $c \prec a[x]$, therefore, since $a[x] \ \eta \ P \sqsubseteq \mathcal{W}$,$c \ \eta \ P$.
We have therefore $P \cap a \cong \mathcal{W} \cap a$ and by lemma 8.26 $a \ \eta \ \mathcal{W}$.
Now, with $Y := y \prec I \to y \ \eta \ \mathcal{W}$, follows $\mathcal{A}^{\mathcal{W}}(Y) \cap I \cong \mathcal{A}^{\mathcal{W}}(\mathcal{W}) \cap I \cong \mathcal{W} \cap I$, $W(\mathcal{W}) \subset Y$, $W(\mathcal{W}) \cap I \subset \mathcal{W} \cap I$. $\mathcal{W} \subset W(\mathcal{W})$ follows by $Ag(\mathcal{W})$.

General Assumption 8.28 *s* *To show, that the ausgezeichnete Mengen and Klassen are closed under the Veblen-function ϕ, the collapsing function D, and that $\mathcal{W}$ is closed under $\lambda x.\Omega_x$, $(x \prec I)$, we need some assumptions about the fundamental sequences, and need to introduce some more functions:*

(a) *We assume functions*

$$\widehat{\phi}, D : N \to N \to N$$

($\widehat{\phi}$ is a version free of fixed points of the Veblen-function, D the collapsing function)

$$Dicom, NF_{D,\mathcal{B}} : N \to N \to \mathcal{B}$$

(Dicom will be the largest $D_I a$, contained in an ordinal. Dicom stands for "D_I-component". $NF_{D,\mathcal{B}}(a,b)$ stands for $D_a b$ is in normal-form and we define $NF_D := \lambda x, y.atom(NF_{D,\mathcal{B}} xy)$)

$$Cr : N \to \mathcal{P}^{dec}(N)$$

($Cr(a)$ will be the set of a-critical ordinals)

$$length : N \to N$$

(the length of an ordinal term),
subsets

$$G, Fi : \mathcal{P}^{dec}(N)$$

(G will be the Gamma ordinals, the a such that $\phi_a 0 = a$, ϕ being the usual version of the Veblen-function (with fixed points). Fi will be the fixed points of $\lambda x.\Omega_x$, that is I and the ordinals $D_I a$)

We assume the following list of properties:

$Fi \subset G \subset A \subset OT,\ R \subset G$

$\forall x, y\ \eta\ OT, n \in N.\widehat{\phi}_x y\ \eta\ OT \wedge (NF_D(x,y) \to D_x y\ \eta\ OT)$

We want, that every element of OT is constructed by elements of smaller length:

$$\begin{aligned}
&\forall x\ \eta\ OT.\exists y, z\ \eta\ OT.length(y), length(z) <_N length(x) \wedge \\
&\quad (NF_{\widehat{+}}(y,z) \wedge z\ \eta\ A \wedge x = y + z) \vee \\
&\quad x = \widehat{\phi}_y z \vee \\
&\quad (NF_D(y,z) \wedge x = D_y z) \vee \\
&\quad x = \Omega_y \vee \\
&\quad x = 0 \vee x = I,
\end{aligned}$$

We demand some properties of the D-normal-form:

$\forall x\ \eta\ R.NF_D(x,0) \wedge \forall y\ \eta\ OT.NF_D(x,y) \to NF_D(x, y+1)$

$\forall x, y\ \eta\ OT.NF_D(x,y) \to D_x y \prec x$

$\forall n \in N.NF_D(\Omega_1, \Omega_{I+1\cdot n})$

$\forall x, y\ \eta\ OT.NF_D(x,y) \to x\ \eta\ R.$

We claim that $Dicom(x)$, is the largest $x \in Fi$ below x:

$\forall x\ \eta\ OT.D_I 0 \preceq x \prec I \to \exists y\ \eta\ OT.NF_D(I,y) \wedge Dicom(x) = D_I y \wedge D_I y \preceq x \prec D_I(y+1)$

And we characterize the critical numbers:

$$\begin{aligned}
&\forall x, y, z\ \eta\ OT.(NF_+(x,y) \to x + y\ \not\eta\ Cr(z)) \wedge \\
&\quad (\widehat{\phi}_x y\ \eta\ Cr(z) \leftrightarrow z \prec x) \wedge (x\ \eta\ G \to (x\ \eta\ Cr(z) \leftrightarrow z \prec x)),
\end{aligned}$$

characterize the additive principal numbers:

$$\forall x, y \,\eta\, OT.(NF_+(x,y) \to x + y \not\eta\, A) \wedge \widehat{\phi}_x y \,\eta\, A \wedge 0 \not\eta\, A \wedge$$
$$(NF_D(x,y) \to D_x y \,\eta\, A) \wedge \Omega_x \,\eta\, A \wedge I \,\eta\, A.$$

the Gamma numbers:

$$\forall x, y \,\eta\, OT.(NF_+(x,y) \to x + y \not\eta\, G) \wedge \widehat{\phi}_x y \not\eta\, G \wedge 0 \not\eta\, G$$
$$\wedge (NF_D(x,y) \to D_x y \,\eta\, G) \wedge (x \neq 0 \to \Omega_x \,\eta\, G) \wedge I \,\eta\, G.$$

the fixed points of $\lambda x.\Omega_x$*:*

$$\forall x, y \,\eta\, OT.(NF_+(x,y) \to x + y \not\eta\, Fi) \wedge \widehat{\phi}_x y \not\eta\, Fi \wedge$$
$$0 \not\eta\, Fi \wedge (NF_D(x,y) \to (D_x y \,\eta\, Fi \leftrightarrow x = I)) \wedge (\Omega_x \,\eta\, Fi \leftrightarrow x \,\eta\, Fi) \wedge I \,\eta\, Fi.$$

and the regular cardinals:

$$\forall x, y \,\eta\, OT.(NF_+(x,y) \to x + y \not\eta\, R) \wedge \widehat{\phi}_x y \not\eta\, R \wedge 0 \not\eta\, G \wedge$$
$$(NF_D(x,y) \to D_x y \not\eta\, R) \wedge$$
$$(\Omega_x \,\eta\, R \leftrightarrow (\exists z \,\eta\, OT.x = z + 1)) \wedge I \,\eta\, R.$$

Further we assume $\omega = \widehat{\phi}_0 1$.

(b) *We assume, that the fundamental sequences are built as follows:*

(A[ ].0)	$\tau(0) = \emptyset$.
(A[ ].1)	*If* $NF_+(b,c)$, $a = b + c$, $b \,\eta\, OT$, $c \neq 0$, *then* $\tau(a) = \tau(c)$, $(b + c)[z] = b + (c[z])$
(A[ ].2)	*Case* $a = \widehat{\phi}_b c$*:*
(A[ ].2.1)	*Case* $b = 0$*:*
(A[ ].2.1.1)	$c = 0 \Rightarrow \tau(a) = 1$, $a[z] = 0$
(A[ ].2.1.2)	$c = c' + 1 \Rightarrow \tau(a) = \omega$, $a[1 \cdot n] = \widehat{\phi}_0(c') \cdot (SSn)$.
(A[ ].2.1.3)	$(c \,\eta\, Cr(b)) \Rightarrow \tau(a) = \omega$, $a[1 \cdot n] = c \cdot (Sn)$.
(A[ ].2.1.4)	$(c \,\eta\, Lim \wedge (c \not\eta\, Cr(b))) \Rightarrow \tau(a) = \tau(c)$, $a[z] = \widehat{\phi}_0(c[z])$
(A[ ].2.2)	*Case* $b = b' + 1$*:*
(A[ ].2.2.1)	$c = 0 \Rightarrow \tau(a) = \omega$, $a[1 \cdot n] = \rho_{SSSn}$ *where* $\rho_0 = 0$, $\rho_{Sn} = \widehat{\phi}_{b'} \rho_n$
(A[ ].2.2.2)	$c = c' + 1 \Rightarrow \tau(a) = \omega$, $a[1 \cdot n] = \rho_n$ *where* $\rho_0 = \widehat{\phi}_b c'$, $\rho_{Sn} = \widehat{\phi}_{b'} \rho_n$
(A[ ].2.2.3)	$(c \,\eta\, Cr(b)) \Rightarrow \tau(a) = \omega$, $a[1 \cdot n] = \rho_{Sn}$, *where* $\rho_0 = c$, $\rho_{Sn} = \widehat{\phi}_{b'} \rho_n$
(A[ ].2.2.4)	$(c \,\eta\, Lim \wedge c \not\eta\, Cr(b)) \Rightarrow \tau(a) = \tau(c)$, $a[z] = \widehat{\phi}_b(c[z])$
(A[ ].2.3)	*Case* $b \,\eta\, Lim$*:*
(A[ ].2.3.1)	$(c = 0 \wedge b \,\eta\, OT \setminus G) \Rightarrow \tau(a) = \tau(b)$, $a[z] = \widehat{\phi}_{b[z]} 0$
(A[ ].2.3.2)	$(c = 0 \wedge b \,\eta\, G) \Rightarrow \tau(a) = \tau(b)$, $a[z] = \widehat{\phi}_{b[z]} b$
(A[ ].2.3.3)	$c = c' + 1 \Rightarrow \tau(a) = \tau(b)$, $a[z] = \widehat{\phi}_{b[z]}(\widehat{\phi}_b c')$
(A[ ].2.3.4)	$(c \,\eta\, Cr(b)) \Rightarrow \tau(a) = \tau(b)$,

$a[z] = \widehat{\phi}_{b[z]}c$

(A[ ].2.3.5) $(c \,\eta\, Lim \wedge c \,\not\eta\, Cr(b)) \Rightarrow \tau(a) = \tau(c)$,

$a[z] = \widehat{\phi}_b(c[z])$

(A[ ].3) *Case* $a = D_b c$:

(A[ ].3.1) *Case* $c = 0$:

(A[ ].3.1.1) $b \neq I \Rightarrow \tau(a) = \omega$,

$a[1 \cdot n] = \rho_{Sn}$, *where* $\rho_0 = b^-$, $\rho_{Sn} = \widehat{\phi}_{\rho_n}0$.

(A[ ].3.1.2) $b = I \Rightarrow \tau(a) = \omega$,

$a[1 \cdot n] = \rho_{SSn}$, *where* $\rho_0 = 0$, $\rho_{Sn} = \Omega_{\rho_n}$.

(A[ ].3.2) *Case* $c = c' + 1$:

(A[ ].3.2.1) $b \neq I \Rightarrow \tau(a) = \omega$,

$a[1 \cdot n] = \rho_n$, *where* $\rho_0 = D_b c'$, $\rho_{Sn} = \widehat{\phi}_{\rho_n}0$.

(A[ ].3.2.2) $b = I \Rightarrow \tau(a) = \omega$,

$a[1 \cdot n] = \rho_n$, *where* $\rho_0 = D_I c'$, $\rho_1 = \Omega_{\rho_0|1}$, $\rho_{SSn} = \Omega_{\rho_{Sn}}$.

(A[ ].3.3) *Case* $c \,\eta\, Lim$, $\tau(c) \prec b$:

$\tau(a) = \tau(c)$, $a[z] = D_b(c[z])$.

(A[ ].3.4) *Case* $c \,\eta\, Lim$, $b \preceq \tau(c)$:

Then $\tau(a) = \omega$, $a[1 \cdot n] = D_b c[\zeta_n]$, *where* ζ_n *is defined by:*

(A[ ].3.4.1) $\tau(c) \neq I \vee b \prec D_I c \vee b = I \Rightarrow$

$\zeta_0 = 0$, $\zeta_{Sn} = D_\pi(c[\zeta_n])$, $\pi := \tau(c)$.

(A[ ].3.4.2) $(\tau(c) = I \wedge D_I c \preceq b \prec I) \Rightarrow$

$\zeta_0 = Dicom(b)$, $\zeta_1 = \Omega_{\zeta_0+1}$, $\zeta_{SSn} = \Omega_{\zeta_{Sn}}$.

(A[ ].4) *Case* $a = \Omega_b$, $b \neq \Omega_b$:

(A[ ].4.1) *Case* $a \,\eta\, R$:

$\tau(a) = a$, $a[z] = a^- + z$

(A[ ].4.2) *Case* $a \,\not\eta\, R$:

$\tau(a) = \tau(b)$, $a[z] = \Omega_{b[z]}$ *or* $a[z] = \Omega_{b[z]+1}$

(A[ ].5) *Case* $a = I$:

$\tau(a) = I$, $a[z] = a^- + z$.

Further we assume, that in (A[ ].1) we have $NF_+(b, c[z])$ *and in the cases (A[ ].3.4) we have* $\forall n \in N.\zeta_n \prec \tau(c)$.

The next lemma is one of the most important, it shows that $\mathcal{W} \cap I$ is closed under Ω..

Lemma 8.29 *(a)* $\forall X \in \mathcal{P}(N).\forall a, b \,\eta\, W(X).\widehat{\phi}_a b \,\eta\, W(X)$.

(b) If $X \in \mathcal{C}l_y(N)$, *then* $\forall a, b \,\eta\, W(X).\widehat{\phi}_a b \,\eta\, W(X)$.

Proof:

(a) Let $Y := \lambda x.\forall y \,\eta\, W(X).\widehat{\phi}_x y \,\eta\, W(X)$,
we show $\mathcal{A}^X(Y) \cap W(X) \subset Y$, and then follows assertion (a).
Assume $x \,\eta\, \mathcal{A}^X(Y) \cap W(X)$, let $Y' := \lambda y.\widehat{\phi}_x y \,\eta\, W(X)$. We show $\mathcal{A}^X(Y') \cap W(X) \subset Y'$, and then follows the assertion.
Assume $y \,\eta\, \mathcal{A}^X(Y') \cap W(X)$, and let $a := \widehat{\phi}_x y$.
Case $y \,\eta\, Lim \setminus Cr(x)$: Then $\forall z \,\eta\, \tau(y)^X = \tau(a)^X.y[z] \,\eta\, Y'$, therefore $\widehat{\phi}_x y[z] \,\eta\, W(X)$, $a \,\eta\, W(X)$.
Assume therefore $y \,\not\eta\, Lim \setminus Cr(x)$:
Case $x = 0$:
Subcase $y = 0$: $a[z] = 0 \,\eta\, W(X)$.

Subcase $y = y' + 1$: $y' \ \eta \ Y'$ since $0 \ \eta \ \tau(y)^X$ and $y \ \eta \ \mathcal{A}^X(Y')$. Therefore $\widehat{\phi}_x y' \cdot n \ \eta \ W(X)$ and $\widehat{\phi}_x y \ \eta \ W(X)$.
Subcase $y \ \eta \ Cr(x)$: $y \ \eta \ W(X)$, therefore $\widehat{\phi}_x y[1{\cdot}n] = y{\cdot}Sn \ \eta \ W(X)$, therefore $\widehat{\phi}_x y \ \eta \ W(X)$.
Case $x = x' + 1$: Then $x' \ \eta \ Y$, $\forall y \ \eta \ W(X).\widehat{\phi}_{x'} y \ \eta \ W(X)$.
Subcase $y = 0$: $\rho_0 = 0 \ \eta \ W(X)$, and if $\rho_n \ \eta \ W(X)$, then $\rho_{Sn} = \widehat{\phi}_{x'}\rho_n \ \eta \ W(X)$, therefore $a \ \eta \ W(X)$.
Subcase $y = y' + 1$: $y' \ \eta \ Y'$, therefore $\rho_0 = \widehat{\phi}_x y' \ \eta \ W(X)$, and if $\rho_n \ \eta \ W(X)$, then $\rho_{Sn} = \widehat{\phi}_{x'}\rho_n \ \eta \ W(X)$, therefore $a \ \eta \ W(X)$.
Subcase $y \ \eta \ Cr(x)$: $\rho_0 = y \ \eta \ W(X)$, and if $\rho_n \ \eta \ W(X)$, then $\rho_{Sn} = \widehat{\phi}_{x'}\rho_n \ \eta \ W(X)$, therefore $a \ \eta \ W(X)$.
Case $x \ \eta \ Lim$:
Subcase $y = 0$, $x \ \not\eta \ G$: $\forall z \ \eta \ \tau(x)^X = \tau(a)^X.\widehat{\phi}_{x[z]}0 \ \eta \ W(X)$, $a \ \eta \ W(X)$.
Subcase $y = 0$, $x \ \eta \ G$: $\forall z \ \eta \ \tau(x)^X = \tau(a)^X.\widehat{\phi}_{x[z]}x \ \eta \ W(X)$, $a \ \eta \ W(X)$.
Subcase $y = y' + 1$: $y' \ \eta \ Y'$, therefore $\widehat{\phi}_x y' \ \eta \ W(X)$,
therefore $\forall z \ \eta \ \tau(x)^X = \tau(a)^X.\widehat{\phi}_{x[z]}(\widehat{\phi}_x y') \ \eta \ W(X)$, $a \ \eta \ W(X)$.
Subcase $y \ \eta \ Cr(x)$: $y \ \eta \ W(X)$, therefore $\forall z \ \eta \ \tau(x)^X = \tau(a)^X.\widehat{\phi}_{x[z]}y \ \eta \ W(X)$, $a \ \eta \ W(X)$.
(b) follows as (a) .

Lemma 8.30 *Let $X : \mathcal{P}(N)$ or $X \in \mathcal{C}l_y(N)$.*

(a) $X \subset W(X) \rightarrow \forall x \ \eta \ X.\Omega_x \in W(X)$.

(b) $(I \ \eta \ X \wedge Ag(X)) \rightarrow \forall a \ \eta \ X \cap I.\Omega_a \ \eta \ X$.

Proof:
(a) Let $Y := \Omega_y \ \eta \ \mathcal{W}$, $Y \in \mathcal{C}l_y(N)$.
We show $\mathcal{A}^X(Y) \cap W(X) \subset Y$, then follows $W(X) \subset Y$.
Assume $a \ \eta \ \mathcal{A}^X(Y) \cap X$.
Case $a = 0$: $\Omega_0 = \omega \ \eta \ W(X)$ by 8.14 (a).
Case $a = a' + 1$: $\Omega_{a'} \ \eta \ W(X)$, by lemma 8.14 (c) $\Omega_a \ \eta \ W(X)$.
Case $\Omega_a = a$: $a \ \eta \ W(X)$.
Case $a \ \eta \ Lim$, $a \neq \Omega_a$: Then
$\forall x \ \eta \ \tau(a)^X.\Omega_a[x] = \Omega_{a[x]} \vee \Omega_a[x] = \Omega_{a[x]+1}$, by assumption $\Omega_{a[x]} \ \eta \ W(X)$ and by 8.14 (c) $\Omega_{a[x]+1} \ \eta \ W(X)$ for $x \ \eta \ \tau(a)^X$, therefore $\Omega_a \ \eta \ \mathcal{A}^X(W(X)) \cong W(X)$.
(b) If $a \in X \cap I$ follows $\Omega_a \in W(X) \cap I$, $\Omega_a \in X$.

Lemma 8.31 *$W(X)$!closed under D Let $X : \mathcal{P}(N)$ or $X \in \mathcal{C}l_y(N)$, $Ag(X)$. Then $\forall a, b \ \eta \ X.NF_D(a, b) \rightarrow D_a b \ \eta \ X$.*

Proof:
We consider the case $X : \mathcal{P}(N)$:
Let $Y := \lambda y.y \ \eta \ X \rightarrow \forall x \ \eta \ X.D_x y \ \eta \ W(X)$,
(in case $X \in \mathcal{C}l_y(N)$ we would choose $Y := y \ \eta \ X \rightarrow \forall x \ \eta \ X.D_x y \ \eta \ W(X)$ for x a fresh variable).
We show $\mathcal{A}^X(Y) \cap W(X) \subset Y$, and then follows $W(X) \subset Y$ and, since $X \sqsubseteq W(X)$ and $D_a b \prec a$ the assertion.
Assume $x \ \eta \ \mathcal{A}^X(Y) \cap X$, $y \ \eta \ X$, and let $a := D_x y$.
Case $y = 0$:
If $x \neq I$, then $x \ \eta \ W(X)$, therefore $\rho_0 = x^- = x[0] \ \eta \ W(X)$ by 8.15 (b) and if $\rho_n \ \eta \ W(X)$ follows $\rho_{Sn} \ \eta \ W(X)$, therefore $a \ \eta \ W(X)$.
If $x = I$, then $\rho_0 \ \eta \ W(X)$ and if $\rho_n \ \eta \ W(X)$, then, since $\rho_n \prec D_I 0 \prec I \ \eta \ X$, $X \sqsubseteq W(X)$, $\rho_n \ \eta \ X$, $\rho_{Sn} \ \eta \ W(X)$ by lemma 8.30 (b), therefore $a \ \eta \ W(X)$.

Case $y = y' + 1$:
If $x \neq I$, then, since $y' \ \eta \ Y$ follows $\rho_0 = D_x y' \ \eta \ W(X)$, and by induction $\rho_n \ \eta \ W(X)$, $a \ \eta \ W(X)$.
If $x = I$, then $\rho_0 \ \eta \ W(X)$ since $y' \ \eta \ Y$, and by induction on $n : N$, since $\rho_n \prec a \prec I$ and, if $\rho_0 \ \eta \ W(X)$ then $\rho_0 + 1 \ \eta \ W(X)$, follows by lemma 8.30 $\rho_n \ \eta \ W(X)$ and therefore $a \ \eta \ W(X)$.
Case $y \ \eta \ Lim$, $\tau(y) \prec x$:
Since $y[z] \ \eta \ Y$ for $z \ \eta \ \tau(y)^X = \tau(a)^X$ follows $a[z] \ \eta \ W(X)$ for these z and therefore $a \ \eta \ W(X)$.
Case $y \ \eta \ Lim$, $x \preceq \tau(y)$:
In this case we always show $\forall n \in N.\zeta_n \ \eta \ W(X)$. Since $\zeta_n \prec y \ \eta \ X$, $\zeta_n \ \eta \ \tau(y)^{\prec}$, follows $\zeta_n \ \eta \ \tau(y)^X$, $y[\zeta_n] \ \eta \ Y$, $a[1 \cdot n] = D_x(y[\zeta_n]) \ \eta \ W(X)$
If $\tau(y) \neq I$ or $x \prec D_I y$ or $x = I$, follows, since by $y \ \eta \ W(X)$ we conclude by 8.15 (d) $\tau(y) \ \eta \ W(X)$, $t := \tau(y) \ \eta \ X$, and, since $y \ \eta \ X$, if $\zeta_n \ \eta \ W(X)$, then $\zeta_n \ \eta \ X$, $x[\zeta_n] \ \eta \ Y$, $\zeta_{Sn} = D_t(a[\zeta_n]) \ \eta \ W(X)$, by induction therefore $\forall n \in N.\zeta_n \ \eta \ W(X)$.
If $\tau(y) = I \wedge D_I y \preceq x \prec I$ we have $Dicom(x) = D_I a$ for some $y \preceq a \ \eta \ OT$, such that $NF_D(I, a)$. Since $D_I(a+1)[0] = D_I a \preceq x \prec D_I(a+1)$, follows by lemma 8.15 (c) $\zeta_0 = Dicom(x) \ \eta \ W(X)$, and because $\zeta_n \prec I$, by an immediate induction $\zeta_n \ \eta \ W(X)$.

Now we want to exhaust the full power of $ML_1^i W_R$, by forming the classes $\mathcal{W}_i$, that are syntactically increasing objects (so we can not define them internal in Martin-Löf's type theory, but only by external meta induction), and where $\mathcal{W}_{Si}$ allows to prove transfinite induction up to $D_{\Omega_1}(\Omega_{I+1 \cdot i})$.

Definition 8.32 $\mathcal{W}_0 := \mathcal{W} \cap I$, $\mathcal{W}_{Si} := W(\mathcal{W}_i) \cap \Omega_{I+1 \cdot Si}$.

Lemma 8.33 *For all* $i \prec \omega$ $Ag(\mathcal{W}_i) \wedge \Omega_{I+1 \cdot i} \ \eta \ \mathcal{W}_{Si} \wedge \mathcal{W}_i \cong \mathcal{W}_{Si} \cap \Omega_{I+1 \cdot i}$.

Proof:
Meta Induction on $i : N$:
$i = 0$: By lemma 8.27 $\mathcal{W}_0 \cong W(\mathcal{W}) \cap I$, and since $\mathcal{W} \cap I \cong \mathcal{W}_0 \cap I$ follows $\mathcal{W}_0 \cong W(\mathcal{W}_0) \cap I \cong \mathcal{W}_1 \cap I$. Therefore $Ag(\mathcal{W}_0) \wedge \mathcal{W}_0 \cong \mathcal{W}_1 \cap \Omega_I$. Further $I^- = 0 \ \eta \ \mathcal{W}_0 \wedge \forall x \ \eta \ \tau(I)^{\mathcal{W}_0}.I[x] = x \ \eta \ \mathcal{W}_0 \subset W(\mathcal{W}_0)$, therefore $\Omega_I = I \ \eta \ W(\mathcal{W}_0) \cap \Omega_{I+1} \cong \mathcal{W}_1$.

$i = Sj$: $\mathcal{W}_j \cong \mathcal{W}_{Sj} \cap \Omega_{I+1 \cdot j}$. Therefore $\mathcal{W}_{Sj} \cong W(\mathcal{W}_j) \cap \Omega_{I+1 \cdot (Sj)} \cong W(\mathcal{W}_{Sj}) \cap \Omega_{I+1 \cdot (Sj)} \cong \mathcal{W}_{SSj} \cap \Omega_{I+1 \cdot (Sj)}$, therefore $Ag(\mathcal{W}_i)$. Further $\Omega^-_{I+1 \cdot (Sj)} = \Omega_{I+1 \cdot j} \ \eta \ \mathcal{W}_{Sj} \wedge \forall x \ \eta \ \tau(\Omega_{I+1 \cdot (Sj)})^{\mathcal{W}_{Sj}}.\Omega_{I+1 \cdot (Sj)}[x] = \Omega_{I+1 \cdot j} + x \ \eta \ \mathcal{W}_{Sj} \subset W(\mathcal{W}_{Sj})$, and we have $\Omega_{I+1 \cdot (Sj)} \ \eta \ W(\mathcal{W}_{Sj}) \cap \Omega_{I+1 \cdot (SSj)} \cong \mathcal{W}_{SSj}$.

Theorem 8.34 *For all* $n \in I\!N$ *we have:*
$ML_1^i W_R \vdash \forall X \in \mathcal{P}(N).(\forall y \in N.(\forall x \in N.x \prec y \rightarrow x \ \eta \ X) \rightarrow y \ \eta \ X) \rightarrow \forall y \in N.y \prec D_{\Omega_1} \Omega_{I+1 \cdot n} \rightarrow y \ \eta \ X$.

Proof:
Assume the premise of the assertion. Then $X : \mathcal{P}(N)$ and $Prog(X)$, therefore by lemma 8.17 (e) and 8.33 $\mathcal{W}_{Sn} \cap \Omega_1 \subset X$. By lemma 8.33 $\Omega_{I+1 \cdot n} \ \eta \ \mathcal{W}_{Sn}$ and $\Omega_1 \ \eta \ \mathcal{W} \cap R \cap I \subset \mathcal{W}_{Sn} \cap R$, therefore by lemma 8.31 $D_{\Omega_1} \Omega_{I+1 \cdot n} \ \eta \ \mathcal{W}_{Sn}$. Since $\mathcal{W}_{Sn} \cap \Omega_1 \sqsubseteq OT$ follows $\forall y \in N.y \prec D_{\Omega_1} \Omega_{I+1 \cdot n} \rightarrow y \ \eta \ X$.

Chapter 9

Properties of the ordinals

In this chapter will now do all the technical work for the introduction of the ordinal denotation system OT. All the work done in this chapter could be carried out as well in HA, that is, we define primitive recursive functions and relations and only need induction over natural numbers in the proofs.
To get unique terms, we will use in this chapter fix point-free versions for the Veblen-function and the enumeration of the infinite cardinals.
The analysis follows well known proof theoretical techniques, which can be found in [BS88], [Buc86], and on which the author did a lot of work in his Diplomarbeit [Set90]. Old versions of this system where studied for instance in [Buc75b] ,[BS76] We start to introduce sets T'', T' and OT (definitions 9.2, 9.7 and 9.10). We use for introducing sums of additive terms a construction $\tilde{+}$, which is the sum of an ordinal with an additive principal number, to avoid the introduction of lists. We define the ordering (definition 9.4) and introduce its properties (9.5). Further we define the set of coefficients $G_\pi a$ needed for the definition of OT. Now we introduce functions like $+$, $\Omega.$, and the set of critical numbers (definition 9.11), show some properties for the lists $Pl(a)$ (in some sense here we switch back to the ordinary introduction of T' — 9.12). We show some easy properties on T'' (9.13, 9.16), closure properties of T' and OT (9.14), some ordering properties of the introduced functions (9.15, 9.17, 9.20), properties of the critical numbers (9.18) and Gamma numbers (9.19). Now we define the fundamental sequences, first a version which we will afterwards modify a little bit for the well-ordering proof (definition 9.21; essentially the difference is that in 9.21 for regular cardinals $a[z] := z$, later we will replace this by $a[z] = a^- + z$ to avoid fixed points of $\Omega.$). We show some easy properties (9.22 and 9.23). In definition 9.24 we introduce a^*, some predecessor relation, having the property that for a modified length lh' we have $lh'(a^*) < lh'(a)$ (9.25), that $s^* \preceq s[\![\tau(s)^-]\!]$ (9.26) $s^* \prec t \preceq s \rightarrow s^* \preceq t^*$ (9.28). We get no new critical numbers, Gamma numbers, regular cardinals or fixed points of $\widehat{\Omega}$ between s^* and s (9.27). If we introduce $t \ll s \Leftrightarrow \exists n \in N.0 <_N n \wedge s^{\overbrace{*\cdots*}^{ntimes}} =_N t$ (definition 9.29), we can show that it exchanges with many functions (9.30), and that, if $\xi \ll \rho$, then $s[\![\xi]\!] \ll s[\![\rho]\!]$ (9.31), and conclude the Bachmann property (lemma 9.32).
To show that OT is closed under forming the fundamental sequences and that $a = sup\{a[\![\xi]\!] | \xi \prec \tau(a) \wedge \xi \;\eta\; OT\}$ for $a \;\eta\; OT$, we motivate on page 109 and introduce in definition 9.33 the relation $\lhd_\xi$, which exchanges with many functions (9.35), allows to deduce properties of $G_\pi a$ (9.36). In lemma 9.37, we show that $s^* \lhd_0 s$, and in lemma 9.38, that $a[\![z]\!] \lhd_z a$. With these properties we can show that we have real fundamental sequences (theorem 9.39) and that OT is closed under fundamental sequences (lemma

9.40).
At the end we modify the fundamental sequences, as stated before. (definition 9.43), prove that essentially we have $a[b[z]] = a[b][z]$ (lemma 9.44) and prove all other properties (lemma 9.45).

Remark 9.1 *We define some Gödel-numbering for the terms:* $\widetilde{+} := \lambda x, y. < \lceil \widetilde{+} \rceil, < x, y >>$, $\widetilde{+} N \to N \to N$ *written as* $a \widetilde{+} b$ *for* $\widetilde{+} ab$, *which corresponds to the sum of an ordinal with an additive principal number in normal-form,*
$\widehat{\phi} := \lambda x, y. < \lceil \widehat{\phi} \rceil, < x, y >>$, $\widehat{\phi} : N \to N \to N$, *written as* $\widehat{\phi}_a b$ *for* $\widehat{\phi} ab$, *which corresponds to the fixed point free version of the Veblen-function*
$\widehat{\Omega} := \lambda x. < \lceil \widehat{\Omega} \rceil, x >$, $\widehat{\Omega} : N \to N$, *written as* $\widehat{\Omega}_a$ *for* $\widehat{\Omega} a$, *which corresponds to the version of the enumeration of the infinite cardinals, which is free of fixed points*
$D := \lambda x, y. < \lceil D \rceil, < x, y >>$, $D : N \to N \to N$, *written* $D_a b$ *for* Dab, *which corresponds to the collapsing function*
$I := < \lceil I \rceil, 0 >$, *(which corresponds to the first weakly inaccessible cardinal)*
$0_{OT} := < \lceil 0 \rceil, 0 >$, *(for 0),*
where $\lceil D \rceil$, $\lceil \widehat{\Omega} \rceil$, $\lceil \widehat{\phi} \rceil$, $\lceil \widetilde{+} \rceil$, $\lceil I \rceil$, $\lceil 0 \rceil$ *are different natural numbers, and* $< \cdot, \cdot >$ *is some primitive recursive pairing operation on* N.

We will introduce three different sets of terms, T'', T', and OT. T'' will be essentially all objects, we can construct by the operations 9.1, demanding only very small normality properties of the terms, without reference to the ordering. T' will be a subset of T'', by restricting the terms to those, which have better properties with respect to $\prec$, which was introduced on T''. OT demands some more conditions for the $D_x y$. It is quite difficult, to prove, that the for $a, b \; \eta \; OT \; a[b] \; \eta \; OT$ ($b \prec \tau(a)$), one of the technically difficult tasks of this chapter, at first hand we only get this closure property for T'.
Besides T'' we define sets Suc'', A'', G'', R'', Fi'', the restriction of them to T' are Suc', A', G', R', Fi', and, restricted to OT, we get Suc, A, G, R, Fi where (where Lim will be the limit ordinals, A the additive principal numbers, G Gamma ordinals, Fi will be the fixed points of $\lambda x.\Omega_x$, that is I and the ordinals $D_I a$, R the regular cardinals).

Definition 9.2 *We give an inductive definition of sets* T'', Suc'', A'', G'', R'' *of terms together with* $length(a)$ *for* $a \; \eta \; T'' \cup Suc'' \cup A'' \cup G'' \cup R'' \cup Fi''$, *such that we can in an immediate way define* $T'', Suc'', A'', G'', R'', Fi''$ *as elements of* $\mathcal{P}^{dec}(N)$ *and length as an element of* $N \to N$.
(T'' *is the set of terms,* Suc'' *the set of successor numbers,* A'' *the additive principal numbers (except* 0_{OT}*),* G'' *the Gamma numbers,* R'' *the regular cardinals,* Fi'' *the set of fixed points of the function* Ω*..)*

- *(T"1)* $0_{OT} \; \eta \; T''$, $length(0_{OT}) := 0$.
- *(T"2)* *If* $a \; \eta \; T'' \setminus \{0_{OT}\}$, $b \; \eta \; A''$, *then* $a \widetilde{+} b \; \eta \; T''$, $a \widetilde{+} 1_{OT} \; \eta \; Suc''$,
  $length(a \widetilde{+} b) := max_N\{length(a), length(b)\} +_N 1$.
- *(T"3)* *If* $a, b \; \eta \; T''$, *then* $\widehat{\phi}_a b \; \eta \; A''$, $\widehat{\phi}_{0_{OT}} 0_{OT} \; \eta \; Suc''$,
  $length(\widehat{\phi}_a b) := max_N\{length(a), length(b)\} +_N 1$.
  and we define $1_{OT} := \widehat{\phi}_{0_{OT}} 0_{OT}$.
- *(T"4)* *If* $b \; \eta \; T''$, *then* $D_I b \; \eta \; Fi''$,
  and if $\pi \; \eta \; R''$ *and* $b \; \eta \; T''$, *then* $D_\pi b \; \eta \; G''$,
  $length(D_\pi b) := max_N\{length(\pi), length(b)\} +_N 1$.
- *(T"5)* *If* $a \; \eta \; Fi'' \cup Suc'' \cup \{0_{OT}\}$, *then* $\widehat{\Omega}_a \; \eta \; R''$,
  if $a \; \eta \; T''$, *then* $\widehat{\Omega}_a \; \eta \; G''$,
  $length(\widehat{\Omega}_a) := length(a) +_N 1$,
- *(T"6)* $I \; \eta \; Fi'' \cup R''$, $length(I) := 0$.

(T"7) $R'' \subset G'' \subset A'' \subset T''$, $Fi'' \subset G''$, $Suc'' \subset T''$.

$Lim'' := T'' \setminus (\{0_{OT}\} \cup Suc'')$.

We will write $0, 1$ *for* $0_{OT}, 1_{OT}$.

Definition 9.3 *We define* $first, last, Alength : N \to N$*, where* $first$ *will be the first,* $last$ *the last additive principal number of the sequence of principal numbers, the ordinal term is built of, and* $Alength$ *be the length of the sequence.*

If $a \not\eta\, T''$ $last(a) := first(a) := Alength(a) := 0$, $last(0_{OT}) := first(0_{OT}) := 0_{OT}$, $Alength(0_{OT}) := 0$, *if* $a\ \eta\ T'' \wedge b\ \eta\ A''$, $last(a\tilde{+}b) := b$, $first(a\tilde{+}b) := first(a)$, $Alength(a\tilde{+}b) := S(Alength(a))$ *and for* $a\ \eta\ A''$ $last(a) := first(a) := a$, $Alength(a) := 1$.

Definition 9.4 *Definition of* $a \prec_{\mathcal{B}} c$ *for* $a, c\ \eta\ T''$*. The definition will be in such a way, that we can define it as an infix written function* $\prec_{\mathcal{B}}: N \to N \to bool$.

We define $a \prec_{\mathcal{B}} b$ *by recursion on* $length(a) +_N length(b)$*, using in the definition* $a \preceq_{\mathcal{B}} b$ *as an abbreviation for* $a \prec_{\mathcal{B}} b \vee_{\mathcal{B}} a =_{N,\mathcal{B}} b$.

$a \prec_{\mathcal{B}} b := ff$, *if* $a \not\eta\, T'' \vee b \not\eta\, T''$.

($\prec$ 1) $(0_{OT} \prec_{\mathcal{B}} c) := \neg_{\mathcal{B}}(0_{OT} =_{N,\mathcal{B}} c)$.
$c \prec_{\mathcal{B}} 0_{OT} := ff$.

($\prec$ 2) $a, c\ \eta\ T''$, $b, d\ \eta\ A''$, *then* $(a\tilde{+}b \prec_{\mathcal{B}} c\tilde{+}d) :=$
$((Alength(a) <_{N,\mathcal{B}} Alength(c)) \wedge_{\mathcal{B}} (a\tilde{+}b \preceq_{\mathcal{B}} c)) \vee_{\mathcal{B}}$
$((Alength(a) =_{N,\mathcal{B}} Alength(c)) \wedge_{\mathcal{B}} (a \prec_{\mathcal{B}} c) \vee_{\mathcal{B}} (a =_{N,\mathcal{B}} c \wedge_{\mathcal{B}} b \prec_{\mathcal{B}} d)) \vee_{\mathcal{B}}$
$((Alength(c) \prec_{\mathcal{B}} Alength(a)) \wedge_{\mathcal{B}} (a \prec_{\mathcal{B}} c\tilde{+}d))$

($\prec$ 3) $a, b\ \eta\ T''$, $c\ \eta\ A'' \setminus \{0_{OT}\}$, *then*
$(a\tilde{+}b \prec_{\mathcal{B}} c) := a \prec_{\mathcal{B}} c$ *and*
$(c \prec_{\mathcal{B}} a\tilde{+}b) := c \preceq_{\mathcal{B}} a$

($\prec$ 4) *If* $a, b, c, d\ \eta\ T''$*, then*
$(\widehat{\phi}_a b \prec_{\mathcal{B}} \widehat{\phi}_c d) :=$
$((a \prec_{\mathcal{B}} c \wedge_{\mathcal{B}} b \prec_{\mathcal{B}} \widehat{\phi}_c d) \vee_{\mathcal{B}} (a =_{N,\mathcal{B}} c \wedge_{\mathcal{B}} b \prec_{\mathcal{B}} d) \vee_{\mathcal{B}}$
$(c \prec_{\mathcal{B}} a \wedge_{\mathcal{B}} \widehat{\phi}_a b \preceq_{\mathcal{B}} d))$.

($\prec$ 5) *If* $a, b\ \eta\ T''$, $c\ \eta\ G''$*, then*
$(\widehat{\phi}_a b \prec_{\mathcal{B}} c) := (a \prec_{\mathcal{B}} c \wedge_{\mathcal{B}} b \prec_{\mathcal{B}} c)$ *and*
$(c \prec_{\mathcal{B}} \widehat{\phi}_a b) := (c \preceq_{\mathcal{B}} a \vee_{\mathcal{B}} c \preceq_{\mathcal{B}} b)$

($\prec$ 6) $\pi, \rho\ \eta\ R''$,$b, d\ \eta\ T''$*, then*
$(D_\pi b \prec_{\mathcal{B}} D_\rho d) :=$
$(\pi =_{N,\mathcal{B}} \rho \wedge_{\mathcal{B}} b \prec_{\mathcal{B}} d) \vee_{\mathcal{B}} (\rho \neq_{N,\mathcal{B}} I \wedge \pi \neq_{N,\mathcal{B}} I \wedge \pi \prec \rho) \vee_{\mathcal{B}}$
$(I =_N \pi \wedge \pi \neq_N I \wedge D_\pi b \prec \rho) \vee_{\mathcal{B}}$
$(\pi \neq_{N,\mathcal{B}} \rho \wedge_{\mathcal{B}} \rho =_{N,\mathcal{B}} I \wedge_{\mathcal{B}} \pi \prec_{\mathcal{B}} D_\rho d)$

($\prec$ 7) *If* $\pi\ \eta\ R''$, $\rho =_N \widehat{\Omega}_c$ *or* $\rho =_N I$, $\pi \neq_N I$, $b\ \eta\ T''$*, then*
$(D_\pi b \prec_{\mathcal{B}} \rho) := \pi \preceq_{\mathcal{B}} \rho$ *and*
$(\rho \prec_{\mathcal{B}} D_\pi b) := \rho \prec_{\mathcal{B}} \pi$.

($\prec$ 8) $b, c\ \eta\ T''$*, then*
$(D_I b \prec_{\mathcal{B}} \widehat{\Omega}_c) := D_I b \preceq_{\mathcal{B}} c$ *and*
$\widehat{\Omega}_c \prec_{\mathcal{B}} D_I b := c \prec_{\mathcal{B}} D_I b$.

($\prec$ 9) $b\ \eta\ T''$*, then*
$D_I b \prec_{\mathcal{B}} I := tt$
$I \prec_{\mathcal{B}} D_I b := ff$

($\prec$ 8) *If* $a, c\ \eta\ T''$*, then*
$(\widehat{\Omega}_a \prec_{\mathcal{B}} \widehat{\Omega}_c) := (a \prec_{\mathcal{B}} c)$

($\prec$ 9) *If* $a\ \eta\ T''$*, then*

$(\widehat{\Omega}_a \prec_{\mathcal{B}} I) := (a \prec_{\mathcal{B}} I)$ *and*
$(I \prec_{\mathcal{B}} \widehat{\Omega}_a) := (I \preceq_{\mathcal{B}} a)$.
Let $\preceq_{\mathcal{B}} := \lambda x, y.x \prec_{\mathcal{B}} y \vee_{\mathcal{B}} x =_{N,\mathcal{B}} y$,
$\prec := \lambda x, y.atom(x \prec_{\mathcal{B}} y)$,
$\preceq := \lambda x, y.atom(x \preceq_{\mathcal{B}} y)$, *all written infix.*

Lemma 9.5 *$(\prec)$ is a linear ordering on T'', that is:*
Assume $a, b, c\, \eta\, T''$. Then:

(a) $\neg(a \prec a)$.

(b) $a \prec b \vee b \prec a \vee a =_N b$.

(c) $a \prec b \rightarrow b \prec c \rightarrow a \prec c$.

Proof: (a) and (b) are easily checked by induction on $length(a)$ and $length(a) +_N length(b)$.
(c): Long and tedious induction on $length(a) +_N length(b) +_N length(c)$:
Case $a =_N 0_{OT}$. Then $c \neq_N 0_{OT}$, $a \prec c$.
Case $b =_N 0_{OT}$ or $c =_N 0_{OT}$ are not possible.
Case $a = d\tilde{+}e$, $b = f\tilde{+}g$, $c = h\tilde{+}k$:
(the case distinction is in the ordering $Alength(f)$, $Alength(h)$, $Alength(d)$):
If $Alength(d), Alength(f) <_N Alength(h)$ follows $a \prec b \preceq h$, therefore $a \prec h$, the assertion.
If $Alength(f) <_N Alength(h) =_N Alength(d)$ follows $d \prec b \prec h$, therefore $a \prec c$.
If $Alength(f) <_N Alength(h) <_N Alength(d)$ follows $d \prec b \prec c$, therefore $a \prec c$.
If $Alength(d) <_N Alength(f) =_N Alength(h)$ follows $a \preceq f \preceq h$, therefore $a \preceq h$, the assertion.
If $Alength(d) =_N Alength(f) =_N Alength(h)$ follows $d \preceq f \preceq h$, therefore $d \prec h$ or $d =_N f =_N h$, $e \prec g \prec k$ and by IH $e \prec k$.
If $Alength(h) =_N Alength(f) \prec Alength(d)$ follows $d \prec b \preceq c$.
If $Alength(d), Alength(h) <_N Alength(f)$ follows $a \preceq f \prec c$.
If $Alength(h) <_N Alength(f) =_N Alength(d)$ follows $d \preceq f \prec c$.
If $Alength(h) <_N Alength(f) <_N Alength(d)$ follows $d \preceq b \prec c$.
Case $\neg(a\, \eta\, A'' \wedge b\, \eta\, A'' \wedge c\, \eta\, A'') \wedge (a\, \eta\, A'' \vee b\, \eta\, A'' \vee c\, \eta\, A'')$:
If $a\, \eta\, A''$, $b =_N f\tilde{+}g$, $c =_N h\tilde{+}k$ follows $a \prec f \preceq h$ or $a \prec f \prec c$ or $a \prec b \prec h$, and the assertion.
If $a =_N d\tilde{+}e$, $b\, \eta\, A''$, $c =_N g\tilde{+}h$, follows $d \prec b \prec c$ or $d \prec b \preceq g$ or $a \prec b \prec g$ and the assertion.
If $a =_N d\tilde{+}e$, $b =_N f\tilde{+}g$, $c\, \eta\, A''$, follows $a \preceq f \prec c$ or $d \preceq f \prec c$ or $d \prec b \prec c$ and the assertion.
If $a, b\, \eta\, A''$, $c = h\tilde{+}k$ follows $a \prec b \prec h$ and the assertion.
If $a, c\, \eta\, A''$, $b = f\tilde{+}g$ follows $a \preceq f \prec c$ and the assertion.
If $b, c\, \eta\, A''$, $a = d\tilde{+}e$ follows $d \prec b \prec c$ and the assertion.
Case $a =_N \widehat{\phi}_d e$, $b =_N \widehat{\phi}_f g$, $c =_N \widehat{\phi}_h k$:
As [BS88], theorem 14.2.
Case $a, b, c\, \eta\, A'' \setminus \{0_{OT}\}$, $\neg(a, b, c\, \eta\, G'')$, $a\, \eta\, G' \vee b\, \eta\, G'' \vee c\, \eta\, G''$:
If $a =_N \widehat{\phi}_d e$, $b =_N \widehat{\phi}_f g$, $c\, \eta\, G''$, follows, if we had $c \preceq d$, $f \prec c \preceq d$, $\widehat{\phi}_d e \prec g \prec c$ by IH, $\widehat{\phi}_d e \prec c$, $d \prec c$. If $d \prec c \preceq e$ follows in case of $d \prec f$ $e \prec \widehat{\phi}_f g \prec c$, $e \prec c$ a contradiction, in case of $d =_N f$, $e \prec g \prec c$, $e \prec c$ a contradiction, and in case of $f \prec d$, $\widehat{\phi}_d e \preceq g \prec c$, $\widehat{\phi}_d e \prec c$, $e \prec c$, again a contradiction. Therefore $d \prec c$, $e \prec c$ and the assertion.

If $a =_N \widehat{\phi}_d e$, $b \,\eta\, G''$, $c =_N \widehat{\phi}_f g$ follows, if $b \preceq f$, $d \prec f$, $e \prec b \prec c$, therefore $e \prec c$ and the assertion. If $f \prec b \preceq g$, follows, if $d \prec f$, $e \prec b \prec c$, therefore $e \prec c$ and the assertion, if $d =_N f$, $e \prec b \preceq g$, $e \prec g$ and $a \prec c$, and if $f \prec d$, $a \prec b \preceq g$, $a \prec g$, $a \prec c$.
If $a \,\eta\, G''$, $b =_N \widehat{\phi}_d e$, $c =_N \widehat{\phi}_f g$, follows, if $d \prec f$, $e \prec c$, therefore, if $a \prec d$, $a \prec f$ and if $a \prec e$, $a \prec c$. If $d =_N f$ follows $e \prec g$, $max_T\{d, e\} \preceq max_T\{f, g\}$. If $f \prec d$, follows $a \prec b \preceq g$, $a \prec g$.
If $a =_N \widehat{\phi}_d e$, $b, c \,\eta\, G''$, follows $max_T\{d, e\} \prec b \prec c$, $max_T\{d, e\} \prec c$.
If $a, c \,\eta\, G''$, $b =_N \widehat{\phi}_d e$, follows $a \preceq max_T\{d, e\} \prec c$.
If $a, b \,\eta\, G''$, $c =_N \widehat{\phi}_d e$, follows $a \prec b \preceq max_T\{d, e\}$.
Case $a =_N D_d e$, $b =_N D_f g$, $c =_N D_h k$:
If $d, f, h \neq_N I$ or $d =_N f =_N h =_N I$ follows $d \preceq f \preceq h$, $d \prec h$ or $e \prec g \prec k$ and the assertion.
If $d =_N I$, $f, h \neq_N I$ follows $a \prec f \preceq h$.
If $f =_N I$, $d, h \neq_N I$ follows $d \prec b \prec h$.
If $h =_N I$, $d, f \neq_N I$ follows $d \preceq f \prec c$, $d \prec c$.
If $d =_N h =_N I$, $f \neq_N I$, follows $a \prec f \prec c$.
If $f =_N h =_N I$, $d \neq_N I$, follows $d \prec b \preceq c$.
If $d =_N f =_N I$, $h \neq_N I$, follows $a \prec b \prec h$.
Case $a =_N D_d e$, $b =_N \widehat{\Omega}_f$ or $b =_N I$, $c =_N D_h k$:
If $d, h \neq_N I$ follows $d \preceq b \prec h$, if $d =_N I$, $h \neq_N I$ follows $a \prec b \prec h$, If $d \neq_N I$, $h =_N I$ follows $d \preceq b \prec c$, and if $d =_N h =_N I$ follows $b \neq_N I$, $a \preceq f \prec b$.
Case $a =_N D_d e$, $b =_N D_e f$ $c =_N \widehat{\Omega}_k$ or $c =_N I$:
If $d, e \neq_N I$ follows $d \preceq e \preceq c$, if $d \neq_N I$, $e =_N I$ follows $d \preceq b \prec c$, $d \prec c$, if $d =_N I$, $e \neq_N I$ follows $a \prec e \preceq c$, and if $d =_N e =_N I$, follows $k \neq_N I$, $a \prec b \prec k$.
Case $a =_N \widehat{\Omega}_d$, $b =_N D_e f$, $c =_N D_g h$:
If $e, g \neq_N I$ follows $a \prec e \preceq g$, if $e =_N I \wedge g \neq_N I$ follows $a \prec b \prec g$, if $e \neq_N I \wedge g =_N I$ follows $a \prec e \prec c$, and if $a =_N g =_N I$ follows $a =_N \widehat{\Omega}_d$, $d \prec b \prec c$, $d \prec c$.
Case $a =_N \widehat{\Omega}_d$ or $a =_N I$, $b =_N \widehat{\Omega}_e$ or $b =_N I$, $c =_N D_g h$:
If $g \neq_N I$ follows $a \prec b \prec g$, and if $g =_N I$ follows $b =_N \widehat{\Omega}_e$, $e \prec I$ therefore $a \neq_N I$, otherwise $I \prec I$, $d \prec e \prec c$, $d \prec c$.
Case $a =_N \widehat{\Omega}_d$ or $a =_N I$, $b =_N D_e f$, $c =_N \widehat{\Omega}_g$ or $c =_N I$:
If $e \neq_N I$ follows $a \prec e \preceq c$, if $e =_N I$ follows $a =_N \widehat{\Omega}_d$, $a \prec c =_N I$ or $a \prec b \preceq g$.
Case $a =_N D_d e$, $b =_N \widehat{\Omega}_e$ or $b =_N I$, $c =_N \widehat{\Omega}_g$ or $c =_N I$:
If $e \neq_N I$ follows $d \preceq b \prec c$, if $d =_N I$ follows, if $c =_N I$, $a \prec c$, if $c =_N \widehat{\Omega}_g$ with $I \preceq g$ $a \prec c$, and if $c =_N \widehat{\Omega}_g$ with $g \prec I$, $b =_N \widehat{\Omega}_e$, $a \prec e \prec g$.
Case $a =_N \widehat{\Omega}_d$, $b =_N \widehat{\Omega}_f$, $c =_N \widehat{\Omega}_h$: $d \prec f \prec h$.
Case $a =_N I$, $b =_N \widehat{\Omega}_f$, $c =_N \widehat{\Omega}_h$: $a \preceq f \prec h$.
Case $a =_N \widehat{\Omega}_d$, $b =_N I$, $c =_N \widehat{\Omega}_h$: $d \prec b \preceq h$.
Case $a =_N \widehat{\Omega}_d$, $b =_N \widehat{\Omega}_f$, $c =_N I$: $d \prec f \prec c$.
The Cases $a =_N I =_N b$ or $b =_N I =_N c$ are not possible.
Case $a =_N I$, $b =_N \widehat{\Omega}_f$, $c =_N I$: $a \preceq f \prec c$.
Definition 9.6 *Let $a \,\eta\, T''$, $M, M' : \mathcal{P}^{fin}(N)$:*

$$
\begin{aligned}
M \preceq_{\mathcal{B}} M' &:= \forall_{\mathcal{B}} x \,\eta\, M \exists_{\mathcal{B}} y \,\eta\, M' (x \preceq_{\mathcal{B}} y),\\
M \prec_{\mathcal{B}} M' &:= \forall_{\mathcal{B}} x \,\eta\, M \exists_{\mathcal{B}} y \,\eta\, M' (x \prec_{\mathcal{B}} y),\\
M \prec_{\mathcal{B}} a &:= M \prec_{\mathcal{B}} \{a\}_{fin},\\
a \preceq_{\mathcal{B}} M &:= \{a\}_{fin} \preceq_{\mathcal{B}} M.
\end{aligned}
$$

$M \preceq M' := atom(M \preceq_{\mathcal{B}} M')$, *similarly for* $\prec$.

Definition 9.7 *We define a set* $T' \subset T''$ *of restricted terms, and define:*

$Suc' := Suc'' \cap T'$, $A' := A'' \cap T'$, $G' := G'' \cap T'$, $R' := R'' \cap T'$. $Lim' := Lim'' \cap T'$, $Fi' := Fi'' \cap T'$.

- (T'1) $0_{OT} \ \eta \ T'$.
- (T'2) *If* $a \ \eta \ T' \setminus \{0_{OT}\}$, $b \ \eta \ A'' \cap T'$, $b \preceq last(a)$, *then* $a \tilde{+} b \ \eta \ T'$.
- (T'3) *If* $a, b \ \eta \ T'$, *then* $\widehat{\phi}_a b \ \eta \ T'$.
- (T'4) *If* $\pi \ \eta \ R'' \cap T'$ *and* $b \ \eta \ T'$, *then* $D_\pi b \ \eta \ T'$.
- (T'5) *If* $a \ \eta \ T'$, *then* $\widehat{\Omega}_a \ \eta \ T'$.
- (T'6) $I \ \eta \ T'$.

We want to construct a system, which has precisely one term for each $\alpha \prec \psi_{\Omega_1}(\Omega_{I+\omega})$. In the set theoretical ordinals it happens, that for regular cardinals κ and certain $\beta' < \beta$, $\psi_\kappa \beta' = \psi_\kappa \beta$. Only if $\beta' \in C_\kappa(\beta')$, $\beta \in C_\kappa(\beta)$ we have $\psi_\kappa \beta' = \psi_\kappa \beta \to \beta = \beta'$. Since we want to assign (and will do this in chapter 10) for every ordinal term t a unique ordinal $o(t)$ by $o(D_a b) = \psi_{o(a)} o(b)$, we need to select those ordinal terms $D_a b$, such that $o(b) \in C_{o(a)}(o(b))$. We will introduce therefore finite sets of ordinals $G_b a$ for $b \ \eta \ R''$, $a \ \eta \ T''$ such that for $a, b, c \ \eta \ T'$ we have $o(c) \in C_{o(b)}(o(a)) \leftrightarrow G_b c \prec a$, and define OT as the subset of T', where only subterms $D_b c$ with $G_b c \prec c$ occur.
$G_b c$ will be the set of the e such that $D_d e$ occurs as a problematic subterm in c. Problematic is $D_d e$, if to conclude $o(c) \in C_{o(b)}(\alpha)$ for an arbitrary α, we need to know $o(D_d e) \in C_{o(b)}(\alpha)$, and additionally $C_{o(b)}(\alpha)$ is only closed under $\rho \mapsto \psi_{o(d)} \rho$ for $\rho \prec \alpha$.

Definition 9.8 *Inductive Definition of* $G_\pi a : \mathcal{P}^{fin}(N)$, $G_\pi a \subset T''$, *such that for* $a \ \eta \ T'$, $\pi \ \eta \ R'$, $G_\pi a \subset T'$, *and we can define it as a function* $G : N \to N \to \mathcal{P}^{fin}(N)$. *The definition is by recursion on* $length(a)$.
$G_\pi a := \emptyset_{fin}$, *if* $\pi \not\eta R' \vee a \not\eta T''$.

- (G1) $G_\pi 0_{OT} := \emptyset$.
- (G2) *If* $a, b \ \eta \ T''$, *then* $G_\pi(a \tilde{+} b) := G_\pi(\widehat{\phi}_a b) := G_\pi a \cup G_\pi b$.
- (G3) *If* $\xi \ \eta \ R''$, $b \ \eta \ T''$, *then*

$$G_\pi D_\xi b := \begin{cases} \{b\} \cup G_\pi \xi \cup G_\pi b, & \text{if } \pi \preceq \xi \neq_N I \vee \\ & \xi =_N I \wedge (\pi \preceq D_I b \vee \pi =_N I), \\ G_\pi \xi & \text{if } \xi \prec \pi =_N I \\ \emptyset, & \text{if } \xi \prec \pi \neq_N I \text{ or} \\ & \xi =_N I \wedge D_I b \prec \pi \prec I. \end{cases}$$

- (G4) *If* $a \ \eta \ T''$, *then* $G_\pi(\widehat{\Omega}_a) := G_\pi a$.
- (G1) $G_\pi I := \emptyset$.

$G^0_\pi a := G_\pi a \cup \{0_{OT}\}$.

Lemma 9.9 *(a)* $a \ \eta \ G_\pi b \to G_\pi a \subset G_\pi b$.

(b) $a \ \eta \ G_\pi b \Rightarrow length(a) <_N length(b)$.

Proof: Easy Induction on $length(a)$.

Definition 9.10 *(a) Inductive Definition of a set of terms* $OT(OT \subset T')$, *such that* OT *can be defined (in an obvious way) as an element* $\mathcal{P}^{dec}(N)$:

- (OT1) $0_{OT} \ \eta \ OT$.
- (OT2) *If* $a \ \eta \ OT \setminus \{0_{OT}\}$, $b \ \eta \ OT \cap A'$, $b \preceq last(a)$, *then* $a \tilde{+} b \ \eta \ OT$
- (OT3) *If* $\pi \ \eta \ R' \cap OT$, $b \ \eta \ OT$, $G_\pi b \prec b$, *then* $D_\pi b \ \eta \ OT$.

(OT4) *If* $a, b\ \eta\ OT$, *then* $\widehat{\phi}_a b\ \eta\ OT$
(OT5) *If* $a\ \eta\ OT$, *then* $\widehat{\Omega}_a\ \eta\ OT$
(OT6) $I\ \eta\ OT$.

The elements of OT *are called* ordinal terms.

(b) $R := R' \cap OT (\cong R'' \cap OT)$,
$Fi := Fi' \cap OT (\cong Fi'' \cap OT)$,
$G := G' \cap OT (\cong G'' \cap OT)$,
$A := A' \cap OT (\cong A'' \cap OT)$,
$Suc := Suc' \cap OT (\cong Suc'' \cap OT)$.

We will now introduce further functions on T'':

Definition 9.11 (a) $\omega := \widehat{\phi}_{0_{OT}} 1_{OT}$, $\omega\ \eta\ OT$.

(b) *Definition of* $\widehat{+} : N \to N \to N$, *written infix. We define* $a \widehat{+} b$ *by recursion on* $length(b)$:
If $a\ \not\eta\ T''$ *or* $b\ \not\eta\ T''$, $a \widehat{+} b := 0_{OT}$.
$a \widehat{+} 0_{OT} := a$, $a \widehat{+} b := a \widetilde{+} b$ *for* $b\ \eta\ A''$, *and* $a \widehat{+} (b \widetilde{+} c) := (a \widehat{+} b) \widetilde{+} c$.
We have $\forall x, y\ \eta\ T''. x \widehat{+} y\ \eta\ T''$.

(c) *Definition of* $+ : N \to N \to N$, *written infix. We define* $a + b$ *by recursion on* $length(a)$, *side recursion on* $length(b)$:
If $a\ \not\eta\ T''$ *or* $b\ \not\eta\ T''$, $a + b := 0_{OT}$.
If $b =_N 0_{OT}$, *then* $a + b := a$.
If $a =_N 0_{OT}$, *then* $a + b := b$.
Let $a, b \neq_N 0_{OT}$.
If $b\ \eta\ A''$, $b \preceq last(a)$, *then* $a + b := a \widetilde{+} b$.
If $b\ \eta\ A''$, $last(a) \prec b$, *then, if* $a\ \eta\ A''$, *then* $a + b := b$, *and if* $a =_N e \widetilde{+} f$, *then* $a + b := e + b$.
If $b =_N e \widetilde{+} f$, $e\ \eta\ T''$, $f\ \eta\ A''$, *then* $a + b := (a + e) \widetilde{+} f$.
We have $\forall x, y\ \eta\ T''. x + y\ \eta\ T''$.

(d) *Definition of* $\cdot : N \to N \to N$, *the multiplication of an ordinal with a natural number, written infix. We define* $a \cdot n$ *by recursion on* $n : N$:
$a \cdot 0 := 0_{OT}$, $a \cdot Sb := (a \cdot b) + a$. *We have* $\forall x\ \eta\ T''. z \in N. x \cdot y\ \eta\ T''$.

(e) *Definition of* $\cdot - \omega, Rest(\cdot, -\omega), pred : N \to N$,

where we write $a - \omega$ *for* $(\cdot - \omega)a$ *(the largest limes-number below* a*)*
$Rest(a, -\omega)$ *for* $Rest(\cdot, -\omega)a$, *(the difference between* a *and* $a - \omega$*)*
$pred(a)$ *for pred* a *(the predecessor of a successor ordinal).*

We define $a - \omega$ $Rest(a, -\omega)$ $pred(a)$ *by recursion on* $length(a)$:

If $a\ \not\eta\ T''$, *then* $a - \omega := pred(a) := 0_{OT}$, $Rest(a, -\omega) := 0$,
$1_{OT} - \omega := 0_{OT}$, $Rest(1_{OT}, -\omega) := 1$, $pred(1_{OT}) := 0_{OT}$,
for $a\ \eta\ T''$, $(a \widetilde{+} 1_{OT}) - \omega := a - \omega$, $Rest(a \widetilde{+} 1_{OT}, -\omega) := S(Rest(a, -\omega))$, $pred(a \widetilde{+} 1_{OT}) := a$,
and if $a\ \eta\ A'' \cup \{0_{OT}\}$ *or* $a =_N b \widetilde{+} c$ *with* $c \neq_N 1_{OT}$, *then* $a - \omega := a$, $Rest(a, -\omega) := 0_{OT}$, $pred(a) := a$.

We prove easily

$$\begin{aligned}\forall x\ \eta\ T''. \quad & x - \omega\ \eta\ T'' \wedge pred(x)\ \eta\ T'' \wedge x =_N a - \omega \widehat{+} (1_{OT} \cdot Rest(x, -\omega)) \wedge \\ & x - \omega\ \eta\ Lim'' \cup \{0_{OT}\} \wedge (x\ \eta\ Suc'' \to x =_N pred(x) \widetilde{+} 1_{OT})).\end{aligned}$$

(f) *Definition of* $\cdot^- : N \to N$, *written as* a^-, *by recursion on* $length(a)$ *(the cardinal-predecessor of a cardinal):*
$a^- := 0_{OT}$ *for* $a \not\eta R'' \cup \{0_{OT}, 1_{OT}, \omega\}$, ${0_{OT}}^- := 0_{OT}$, ${1_{OT}}^- := 0_{OT}$, $\omega^- := 0_{OT}$, $\widehat{\Omega}^-_{0_{OT}} := \omega$, *if* $a \ \eta \ Suc''$, *then* $\widehat{\Omega}^-_a := \widehat{\Omega}_{Pred(a)}$, *if* $a \ \eta \ Fi''$, *then* $\widehat{\Omega}^-_a := a$, *and* $I^- := 0_{OT}$. *We have* $\forall x \ \eta \ T''.x^- \ \eta \ T''$.

(g) *Definition of* $\Omega. : N \to N$, *written as* Ω_a *for* $\Omega.a$.

$$\Omega_a := \begin{cases} a & \text{if } a \ \eta \ Fi'' \\ \widehat{\Omega}_{pred(a)} & \text{if } a - \omega \ \eta \ Fi'' \text{ and } a \not\eta Fi'' \\ 0_{OT} & \text{if } a \not\eta T'' \\ \widehat{\Omega}_a & \text{otherwise.} \end{cases}$$

We have $\forall x \ \eta \ T''.\Omega_x \ \eta \ T''$.

(h) $max_T := \lambda x, y.\ \underline{if}\ x \prec y\ \underline{then}\ y\ \underline{else}\ x$,
$min_T := \lambda x, y.\ \underline{if}\ x \prec y\ \underline{then}\ x\ \underline{else}\ x$,
$max_T, min_T : N \to N \to N$, *and we write* $max_T\{a, b\}, min_T\{a, b\}$ *for* $max_T ab$, $min_T ab$.
We have $\forall x, y \ \eta \ T''.max_T\{x, y\}, min_T\{x, y\} \ \eta \ T''$.

(i) *Definition of* $NF_{+,\mathcal{B}} : N \to N \to \mathcal{B}$:
$NF_{+,\mathcal{B}} := \lambda x, y.x =_{N,\mathcal{B}} 0_{OT} \vee_{\mathcal{B}} first(y) \preceq_{\mathcal{B}} last(x)$, $NF_+ := \lambda x, y.atom(NF_{+,\mathcal{B}}(x, y))$, $NF_+ : N \to N \to U$
We write $NF_{+,\mathcal{B}}(a, b)$, *for* $NF_{+,\mathcal{B}}ab$, *similarly of* NF_+.

(j) *Definition of* $Pl : N \to Nlist$, *which is the list of the additive principal numbers, an ordinal is sum of:*

We define $Pl(a)$ *by recursion on* $length(a)$*:*

$Pl(a) := nil$, *if* $a \not\eta T''$ *or* $a =_N 0_{OT}$, $Pl(a \tilde{+} b) := append(Pl(a), Pl(b))$, *and for* $a \ \eta \ A''$ $Pl(a) := cons(a, nil)$.

(k) *Definition of* $Cr : N \to \mathcal{P}^{dec}(N)$, *where* $Cr(a)$ *should be the set of a-critical ordinals. We define* $Cr\ ab$ *by recursion on* $length(a) +_N length(b)$*:*

$Cr\ ab$*:= false if* $a \not\eta T'' \vee b \not\eta T''$, *and for* $a, b, c \ \eta \ T''$, $Cr\ a0_{OT} := ff$, $Cr\ a(b \tilde{+} c) := ff$, $Cr\ a\widehat{\phi}_b c := a \prec_{\mathcal{B}} b$, *and for* $b \ \eta \ G''$, $Cr\ ab := a \prec_{\mathcal{B}} b$.

We write $Cr(a)$ *for* $Cr\ a$.

(l) *Definition of* $Dicom : N \to N$ *such that* $Dicom(a) \ \eta \ T'$ *for* $a \ \eta \ T'$, $Dicom(a) := 0_{OT}$ *otherwise.* $Dicom(a)$ *should be the largest component* $D_I c$ *contained in* a
$Dicom(a) := 0_{OT}$ *if* $a \not\eta T''$,
$Dicom(0_{OT}) := 0_{OT}$, $Dicom(a \tilde{+} b) := Dicom(\widehat{\phi}_a b) :=$
$max_T\{Dicom(a), Dicom(b)\}$.

$Dicom(D_a b) := Dicom(a)$ *for* $a \neq_N I$, $Dicom(D_I a) := D_I a$, $Dicom(\widehat{\Omega}_a) := Dicom(a)$, $Dicom(I) := I$.

(m) *Definition of* $NF_{D,\mathcal{B}} : N \to N \to \mathcal{B}$ *written as* $NF_{D,\mathcal{B}}(a, b)$*:*
$NF_{D,\mathcal{B}}(a, b) := a \ \eta \ R'' \wedge b \ \eta \ T'' \wedge G_a b \prec_{\mathcal{B}} b$.

We will now characterize the ordering of ordinals, which are not principal additive numbers by the lexicographic ordering of *Plist*.

Lemma 9.12 *(a)* $\forall x \ \eta \ T''.Alength(x) =_N lh(Pl(x)) \wedge (x \neq_N 0_{OT} \to (first(x) =_N (Pl(a))_0 \wedge last(x) =_N (Pl(x))_{Alength(x) \dot{-} 1})) \wedge \forall i <_N lh(Pl(x)).(Pl(x))_i \in A''$.

(b) $\forall x\, \eta\, T''.x\, \eta\, T' \leftrightarrow$
$((\forall i \in N.Si <_N Alength(x) \rightarrow (x)_{Si} \preceq (x)_i) \wedge \forall i <_N Alength(x).(x)_i\, \eta\, T'$.

(c) $\forall x, y\, \eta\, T''.Pl(x\widehat{+}y) \cong_{Nlist} append(Pl(x), Pl(y))$.

(d) $\forall x\, \eta\, T'.x\, \eta\, OT \leftrightarrow \forall i \prec Alength(x).(Pl(x))_i\, \eta\, OT$.

(e) If $a, b\, \eta\, T''$, $la := Pl(a)$, $lb := Pl(b)$, then
$a \prec b \leftrightarrow (\exists i \leq_N min_N\{lh(la), lh(lb)\}.(la)_i \prec (lb)_i \wedge$
$\forall j <_N i.(la)_j =_N (lb)_j) \vee (lh(la) <_N lh(lb) \wedge \forall i <_N lh(la).(la)_i =_N (lb)_i)$.

(f) If $a, b\, \eta\, T''$, $la \cong_{Nlist} Pl(a)$, $lb \cong_{Nlist} Pl(b)$, then
$a =_N b \leftrightarrow (lh(la) =_N lh(lb) \wedge \forall i <_N lh(la).(la)_i =_N (lb)_i)$.

(g) Let

$$\begin{aligned} Q(a,b,i) := \; & i \leq_N Alength(a) \wedge (\forall j <_N Alength(a).i \leq_N j \rightarrow (Pl(a))_0 \prec (Pl(b))_i) \wedge \\ & (i \neq 0_{OT} \rightarrow (Pl(b))_{i \dot{-} 1} \preceq (Pl(a))_0) \end{aligned}$$

Then $\forall a, b\, \eta\, T''.b \neq 0_{OT} \rightarrow (\exists i \in N.Q(a,b,i)) \wedge$
$(\forall i, j \in N.Q(a,b,i) \wedge Q(a,b,j) \rightarrow i =_N j) \wedge$
$\forall i.Q(a,b,i) \rightarrow Pl(a+b) \cong append(Sublist(Pl(a), i), Pl(b))$

(h) $\forall x \in T''.G_\pi x = \bigcup_{i=1}^{Alength(x)} G_\pi Pl((x)_i)$.

(or the formalization of this theorem).

Proof:
(a) follows by an easy induction on $Alength(x)$
(b) Induction on $Alength(x)$:
If $x =_N 0_{OT}$, then $x\, \eta\, T'$ and $Pl(x) \cong_{Nlist} nil$.
If $x\, \eta\, A''$, then $Pl(x) \cong_{Nlist} cons(x, nil)$, $x\, \eta\, T' \leftrightarrow (Pl(x))_0\, \eta\, T'$ and we have the assertion.
If $x =_N b\widetilde{+}c$, then $x\, \eta\, T' \leftrightarrow (b\, \eta\, T' \wedge c\, \eta\, A' \wedge c \preceq last(a) =_N (a)_{Alength(a) \dot{-} 1}) \leftrightarrow (((\forall i \in N.Si <_N Alength(b) \rightarrow (b)_{Si} \preceq (b)_i) \wedge \forall i <_N Alength(b).(b)_i\, \eta\, T') \wedge c\, \eta\, A' \wedge c \preceq (b)_{Alength(b) \dot{-} 1}) \leftrightarrow ((\forall i \in N.Si <_N Alength(a) \rightarrow (a)_{Si} \preceq (a)_i) \wedge \forall i <_N Alength(a).(a)_i\, \eta\, T')$.
(d) follows in a similar way.
(c) follows by an easy induction on $Alength(y)$
(e), (f): Let
$Less(a,b) := (\exists i <_N min_N\{lh(Pl(a)), lh(Pl(b))\}.(Pl(a))_i \prec (Pl(b))_i \wedge$
$\forall j <_N i.(Pl(a))_j =_N (Pl(b))_j) \vee$
$(lh(Pl(a)) <_N lh(Pl(b)) \wedge \forall i <_N lh(Pl(a)).(Pl(a))_i =_N (Pl(b))_i)$,
$Eq(a,b) := (lh(Pl(a)) =_N lh(Pl(b)) \wedge \forall i <_N lh(Pl(a)).(Pl(a))_i =_N (Pl(b))_i)$,
and $m := max_N\{lh(la), lh(lb)\}$.
We show $\forall a, b\, \eta\, T''.(a \prec b \leftrightarrow Less(a,b)) \wedge (a = b \leftrightarrow Eq(a,b))$ by induction on $Alength(a) + Alength(b)$. Assume $a, b\, \eta\, T''$, and let $la := Pl(a)$, $lb := Pl(b)$.
If $b =_N 0_{OT}$, follows $\neg(a \prec b)$, $\neg Less(a,b)$, $a =_N b \leftrightarrow b =_N 0_{OT} \leftrightarrow lh(lb) = 0 \leftrightarrow Eq(a,b)$.
If $a =_N 0_{OT}$, $b \neq_N 0_{OT}$ follows $a \prec b$, and $lh(la) <_N lh(lb) \wedge \forall i <_N lh(la).(la)_i =_N (lb)_i$, $Less(a,b)$, further $a \neq_N b$, $lh(la) \neq_N lh(lb)$, $\neg(Eq(a,b))$.
Case $a =_N c\widetilde{+}d$, $b =_N e\widetilde{+}f$. If $Alength(c) <_N Alength(e)$ follows $m =_N Alength(a) =_N min_N\{Alength(a), Alength(e)\}$, $a \prec b \leftrightarrow a \preceq e \leftrightarrow (Less(a,e) \vee Eq(a,e)) \leftrightarrow Less(a,b)$, and $a \neq_N b$, $\neg(Eq(a,b)$. If $Alength(c) =_N Alength(e)$ follows $m =_N Alength(a) =_N$

$S\ min\{Alength(c), Alength(e)\}$, $a \prec b \leftrightarrow (c \prec e \vee (c =_N e \wedge d \prec f)) \leftrightarrow (Less(c,e) \vee (Eq(c,e) \wedge d \prec f))) \leftrightarrow Less(a,b)$. Further $a =_N b \leftrightarrow (c =_N e \wedge d =_N f) \leftrightarrow (Eq(c,e) \wedge d =_N f) \leftrightarrow Eq(a,b)$. If $Alength(e) <_N Alength(c)$ follows the assertion symmetrically to the case $Alength(c) <_N Alength(e)$.
Case $a =_N c\tilde{+}d$, $b\ \eta\ A'$. Then $lh(lb) = 1 <_N lh(la)$, $m = 1$, $a \prec b \leftrightarrow c \prec b \leftrightarrow Less(c,b) \leftrightarrow Less(a,b)$, $a \neq_N b$, $\neg(Eq(a,b))$.
Case $a\ \eta\ A'$, $b =_N c\tilde{+}d$. Then $lh(la) = 1 <_N lh(lb)$, $m = 1$, $a \prec b \leftrightarrow a \preceq c \leftrightarrow (Less(a,c) \vee Eq(a,c)) \leftrightarrow Less(a,b)$, $a \neq_N b$, $\neg(Eq(a,b))$.
Case $a, b\ \eta\ A'$. Then $a \prec b \leftrightarrow Less(a,b)$, $a =_N b \leftrightarrow Eq(a,b)$.
(g): The existence and uniqueness of i is easy. We show $Q(a,b,i) \wedge b \neq_N 0_{OT} \to Pl(a+b) \cong_{Nlist} append(Sublist(a,i), b)$ by main induction on $Alength(a)$, side induction on $Alength(b)$:
Case $a =_N 0_{OT}$. Then $i =_N 0$ and we have the assertion.
Case $a \neq_N 0_{OT}$:
Case $b\ \eta\ A''$:
If $b \preceq last(a)$, then $i =_N Alength(a)$, $a + b =_N a\tilde{+}b$, $Pl(a+b) = append(Pl(a), Pl(b))$.
If $a\ \eta\ A''$, $a \prec b$, then $i =_N 0$, $Sublist(a,i) \cong_{Nlist} nil$, $a + b =_N b$.
If $a =_N c\tilde{+}d$, $d \prec b$, then $Q(c,b,i)$, $a + b =_N c + b$ and we have the assertion.
Case $b =_N e\tilde{+}f$. Then $Q(a,e,i)$, $Pl(a+b) \cong_{Nlist} Pl((a+e)\tilde{+}f) \cong_{Nlist}$

$$append(append(Sublist(Pl(a),i), Pl(e)), Pl(f)) \cong_{Nlist}$$
$$append(Sublist(Pl(a),i), Pl(b))$$

(h): Easy induction on $Alength(x)$.

Lemma 9.13 *Let $a, b, c, d\ \eta\ T'$.*

(a) $a =_N 0_{OT} \vee (\exists a'\ \eta\ T'.a =_N a'\tilde{+}1_{OT}) \vee a\ \eta\ Lim'$

(b) $a =_N 0_{OT} \to \neg(a\ \eta\ Suc' \vee a\ \eta\ Lim')$ *and*
$a\ \eta\ Suc' \to a\ \not\eta\ Lim'$.

(c) $a =_N a'\tilde{+}1_{OT} \leftrightarrow (a\ \eta\ Suc' \wedge a' =_N pred(a))$

(d) $a, b\ \eta\ T''$, *then*
$b \neq_N 0_{OT} \to last(a\hat{+}b) =_N last(b)$,
$a \neq_N 0_{OT} \to first(a\hat{+}b) =_N first(a)$,
$Alength(a\hat{+}b) =_N Alength(a) +_N Alength(b)$.

(e) $\forall x, y, z\ \eta\ T''.(x\hat{+}y)\hat{+}z =_N x\hat{+}(y\hat{+}z)$

(f) $\forall x, y, z\ \eta\ T'.(x+y)+z =_N x+(y+z)$.

(g) $\forall x, y\ \eta\ T'.x\hat{+}y\ \eta\ T' \leftrightarrow (x =_N 0_{OT} \wedge first(y) \preceq last(x))$ $(\leftrightarrow NF_+(x,y))$.
$\forall x, y\ \eta\ T'.NF_+(x,y) \to x + y =_N x\hat{+}y$.

(h) $(a, b, c\ \eta\ T' \wedge c \prec b) \to (NF_+(a,b) \to NF_+(a,c)) \wedge a + b \prec a + c$.

Proof: (a) - (e) are easy.
(f) as usual using 9.12 (b) and (g).
(g): Use 9.12 (b) and (c).
(h): $first(c) \preceq first(b)$, the second assertion follows as usually.

Lemma 9.14 *(a)* $\forall x,y\ \eta\ T'.\forall z \in N.x+y, x\cdot z, x-\omega, pred(x), x^{-}, \Omega_x, max_T\{x,\}, min_T\{x,y\}, Dicom(x)\ \eta\ T' \land (y\ \eta\ R' \to G_y x \subset T') \land (NF_+(x,y) \to x\widehat{+}y\ \eta\ T')$.

(b) $\forall x,y\ \eta\ OT.\forall z \in N.x+y, x\cdot z, x-\omega, pred(x), x^{-}, \Omega_x, max_T\{x,\}, min_T\{x,y\}, Dicom(x)\ \eta\ OT \land (y\ \eta\ R \to G_y x \subset R) \land (NF_+(x,y) \to x\widehat{+}y\ \eta\ OT)$.

Proof:
(a): for $+$ and $\widehat{+}$ the assertion follows by 9.12 (b), (c) and (g). The other assertions are easy.
(b) Use 9.12 (d), 9.12 (c), 9.12 (g), (a) for $+$, $\widehat{+}$. For the other functions the assertion is easy.

Lemma 9.15 *(a) If* $a, a', b, b'\ \eta\ T''$, $Alength(a) =_N Alength(a')$, *then* $a\widehat{+}b \prec a'\widehat{+}b' \leftrightarrow (a \prec a' \lor (a =_N a' \land b \prec b')$, *and* $a\widehat{+}b =_N a'\widehat{+}b' \leftrightarrow (a =_N a' \land b =_N b')$.

(b) If $a, b, b', c\ \eta\ T''$, $a\widehat{+}b \preceq c \preceq a\widehat{+}b'$, *then* $c =_N a\widehat{+}c'$ *for some* $c'\ \eta\ T''$ *such that* $b \preceq c' \preceq b'$.

(c) $a\widehat{+}b\ \eta\ T' \to a \preceq a\widehat{+}b \land b \preceq a\widehat{+}b$.

(d) $\forall x,y \in T'', \pi \in R''.G_\pi y \subset G_\pi(x+y) \subset G_\pi x \cup G_\pi y \land G_\pi(x\widehat{+}y) = G_\pi x \cup G_\pi y$

(e) $\forall \pi\ \eta\ R'.\pi \neq_N I \to length(\pi^-) <_N length(\pi)$.

(f) $\forall a,b\ \eta\ T'.a \prec \widehat{\phi}_a b \land b \prec \widehat{\phi}_a b$.

(g) $\forall a\ \eta\ T'.a \prec \widehat{\Omega}_a$.

(h) $\forall \pi\ \eta\ R', a\ \eta\ T'.a \prec \pi \to a \prec D_\pi a$.

Proof:
(a), (b), (c): Use 9.12, and the common known properties of the lexicographic ordering.
(d) follows by 9.12 (c), (g) and (h).
(e) Trivial. (f) (i) We prove $b \prec \widehat{\phi}_a b$ by induction on $length(b)$:
If $b =_N 0_{OT}$ this is trivial, if $b =_N c\widetilde{+}d$ follows $c \prec \widehat{\phi}_a c \prec \widehat{\phi}_a b$ by IH and (c).
Case $b =_N \widehat{\phi}_c d$: If $c \prec a$ follows $d \prec \widehat{\phi}_a d \prec \widehat{\phi}_a(\widehat{\phi}_c d)$, therefore the assertion. If $c =_N a$ follows $d \prec \widehat{\phi}_c d =_N b$ and the assertion. If $a \prec c$ follows $\widehat{\phi}_c d \preceq b$, $\widehat{\phi}_c d \prec \widehat{\phi}_a b$.
If $b\ \eta\ G'$ follows the assertion by definition.
(ii) Now we prove $a \prec \widehat{\phi}_a b$ by induction on $length(a)$:
If $a =_N 0_{OT}$ this follows by definition, if $a =_N c\widetilde{+}d$ follows $c \prec \widehat{\phi}_c b \prec \widehat{\phi}_a b$ by IH, therefore $a \prec \widehat{\phi}_a b$. If $a =_N \widehat{\phi}_c d$ follows by IH $c \prec a$, and since $d \prec a$, $b \prec \widehat{\phi}_a b$ by (i), $d \prec \widehat{\phi}_d b \prec \widehat{\phi}_a b$.
If $a\ \eta\ G'$ follows the assertion by definition.
(g) By Induction on $length(a)$:
The cases $a =_N 0_{OT}$, $a =_N I$ follow by definition.
If $a =_N b\widetilde{+}c$, follows by IH $b \prec \widehat{\Omega}_b$, therefore $a \prec \widehat{\Omega}_b \prec \widehat{\Omega}_a$.
If $a =_N \widehat{\phi}_b c$, follows $b \prec \widehat{\Omega}_b \prec \widehat{\Omega}_{\widehat{\phi}_b c}$, $c \prec \widehat{\Omega}_c \prec \widehat{\Omega}_{\widehat{\phi}_b c}$, therefore $\widehat{\phi}_b c \prec \widehat{\Omega}_a$.
If $a =_N D_{\widehat{\Omega}_{b\widetilde{+}1_{OT}}} c$, follows $\widehat{\Omega}_b \prec \widehat{\Omega}_{\widehat{\Omega}_b}$, $\widehat{\Omega}_{b\widetilde{+}1_{OT}} \prec \widehat{\Omega}_{\widehat{\Omega}_b}$, $a \prec \widehat{\Omega}_{b\widetilde{+}1_{OT}} \prec \widehat{\Omega}_{\widehat{\Omega}_b} \prec \widehat{\Omega}_a$.
If $a =_N D_{\widehat{\Omega}_b} c$, $b =_N D_I d$ or $b =_N I$, follows $D_{\widehat{\Omega}_b} c \prec \widehat{\Omega}_b \prec \widehat{\Omega}_{D_{\widehat{\Omega}_b} c}$.
If $a =_N D_{\widehat{\Omega}_{0_{OT}}} c$, follows $D_{\widehat{\Omega}_{0_{OT}}} c \prec \widehat{\Omega}_{0_{OT}} \prec \widehat{\Omega}_{D_{\widehat{\Omega}_{0_{OT}}} c}$.
If $a =_N D_I b$, follows $a \prec \widehat{\Omega}_a$ by definition.
If $a =_N \widehat{\Omega}_b$ follows by IH $b \prec \widehat{\Omega}_b$, therefore $a \prec \widehat{\Omega}_a$.
(h) If $a =_N 0_{OT}$ this is trivial, if $a =_N I$ follows $a \preceq \pi^- \preceq D_\pi a$.

If $a =_N b\tilde{+}c$ follows $b \prec D_\pi b \preceq D_\pi a$, therefore $a \preceq D_\pi a$, if $a =_N \widehat{\phi}_b c$ follows $b, c \prec D_\pi max_T\{b, c\} \preceq D_\pi a$. If $a =_N D_\rho b$ follows if $\rho \prec \pi$ and $\pi \neq_N I$, $a \prec \pi^- \preceq D_\pi a$, if $\rho \prec \pi =_N I$ follows by IH, since $length(\rho^-) <_N length(\rho)$, $\rho^- \prec D_\pi \rho^- \preceq D_\pi a$, therefore $a \prec \rho \prec D_\pi a$, further if $\pi \prec I =_N \rho$ follows from $a \prec \pi$ $a \preceq \pi^- \prec D_\pi a$, and if $\rho =_N \pi$ follows by IH $b \prec D_\pi b$, $a =_N D_\pi b \prec D_\pi a$.
If $a =_N \widehat{\Omega}_b$ follows, in case of $\pi =_N I$ $b \prec D_\pi b \preceq D_\pi a$, $a \prec D_\pi a$, and if $\pi \neq_N I$, $a \preceq \pi^- \prec D_\pi a$.

Lemma 9.16 *(a)* $\forall x \, \eta \, T''.\neg(x \prec 0_{OT})$.

(b) $\forall x \, \eta \, T''.x \prec 1_{OT} \leftrightarrow x =_N 0_{OT}$.

(c) $\forall x \, \eta \, T'.x \prec \omega \rightarrow \exists n \in N.x =_N 1_{OT} \cdot n$.

(d) $a \neq_N 0_{OT}$, $a \, \eta \, T' \rightarrow 1_{OT} \preceq a$.

Proof: All by 9.12 (b) and (e) and $\forall x \, \eta \, A'.x =_N 0_{OT} \vee x =_N 1_{OT} \vee x =_N \omega \vee \omega \prec x$.

Lemma 9.17 *(a)* $\forall x \, \eta \, R' \setminus \{I\}.\forall y \, \eta \, (R' \cup Fi').y \prec x \rightarrow y \preceq x^-$.

(b) $\forall x \, \eta \, R' \setminus \{I\}.\forall y \, \eta \, (R' \cup Fi').x^- \preceq y \preceq x \rightarrow y \, \eta \, \{x^-, x\}$.

(c) $\forall x \, \eta \, R'.\forall y \, \eta \, A'.y \prec x \rightarrow x^- + y \prec x$.

Proof:
(a) $x =_N \widehat{\Omega}_{0_{OT}}$ is not possible. If $x =_N \widehat{\Omega}_{a\tilde{+}1_{OT}}$, $y =_N \widehat{\Omega}_e$ follows $e \prec a\tilde{+}1_{OT}$ therefore $e \preceq a$, $y \preceq \widehat{\Omega}_a =_N x^-$. If in the same case $y =_N I$, follows $I \preceq a\tilde{+}1_{OT}$, $I \preceq a$, $I \prec \widehat{\Omega}_a$, and if $y =_N D_I d$ follows $y \prec a\tilde{+}1_{OT}$, $y \preceq a \prec \widehat{\Omega}_a$. If $x =_N \widehat{\Omega}_a$ $x^- =_N a$, $x =_N D_I f$ or $x =_N I$, $y =_N \widehat{\Omega}_e$, follows by $e \prec a$, $\widehat{\Omega}_e \prec a$, and if $y =_N I$ or $y =_N D_I d$, follows $y \preceq a$.
(b) by (a).
(c) If $y \, \eta \, A'$, $y \preceq x^-$ this follows by $x^- \prec x$, if $x^- \prec y \, \eta \, A'$ this is trivial, and if $y = y_1 \tilde{+} y_2$ follows $x^- + y_1 \prec x$ and the assertion.

Lemma 9.18 *Assume* $a, b, c \, \eta \, T'$.

(a) $a \, \eta \, Cr(b) \rightarrow b \prec a$.

(b) $a \, \eta \, Cr(b) \rightarrow ((a \prec \widehat{\phi}_b c \leftrightarrow a \preceq c) \wedge (\widehat{\phi}_b c \preceq a \leftrightarrow c \prec a))$.

(c) $b \, \eta \, Cr(a) \rightarrow \widehat{\phi}_{a\tilde{+}1_{OT}} 0_{OT} \preceq b$.

(d) $\widehat{\phi}_a b \prec c \preceq \widehat{\phi}_a (b\tilde{+}1_{OT}) \rightarrow c \, \not\eta \, Cr(a) \cup G'$.

(e) $(b \, \eta \, Cr(a) \wedge b \prec c \preceq \widehat{\phi}_a b) \rightarrow c \, \not\eta \, Cr(a) \cup G'$.

(f) $a \prec \widehat{\phi}_b 0_{OT} \rightarrow a \, \not\eta \, Cr(b)$.

Proof:
(a) If $a =_N \widehat{\phi}_c d$ with $b \prec c$ follows $b \prec c \prec a$, and if $a \, \eta \, G'$ we have $b \prec a$.
(b) If $a =_N \widehat{\phi}_f g$ with $b \prec f$ follows the assertion by definition, and if $b \prec a =_N G'$ follows in the first part of the assertion $a \prec \widehat{\phi}_b c \leftrightarrow a \preceq max_T\{b, c\} \leftrightarrow a \preceq c$ and in the second part $\widehat{\phi}_b c \preceq a \leftrightarrow max_T\{b, c\} \prec a \leftrightarrow c \prec a$.
(c) If $a =_N \widehat{\phi}_c d$ with $a \prec c$, follows $a\tilde{+}1_{OT} \preceq c$, $\widehat{\phi}_{a\tilde{+}1_{OT}} 0_{OT} \preceq a$, and if $a \prec b \, \eta \, G'$ follows $max_T\{a\tilde{+}1_{OT}, 0_{OT}\} \prec b$, $\widehat{\phi}_{a\tilde{+}1_{OT}} 0_{OT} \prec b$.
(d) Assume $c \, \eta \, Cr(a)$. By (b) follows $b \prec c \preceq b\tilde{+}1_{OT}$, $c =_N b\tilde{+}1_{OT}$ a contradiction. If $c \, \eta \, G'$ follows $c \, \eta \, Cr(a)$.

(e) Assume $c\ \eta\ Cr(a)$. By (b) follows $b \prec c \preceq b\tilde{+}1_{OT}$, $c =_N b\tilde{+}1_{OT}$ a contradiction. If $c\ \eta\ G'$ follows $c\ \eta\ Cr(a)$.
(f) Assume $a\ \eta\ Cr(b)$. By (b) follows $a \preceq 0_{OT}$.

Lemma 9.19 *(a)* $\forall x\ \eta\ R' \setminus \{I\}.\forall z\ \eta\ T'.x^- \prec z \prec D_x 0_{OT} \rightarrow z \not\eta\ G'$.

(b) $\forall x\ \eta\ R' \setminus \{I\}.\forall y, z\ \eta\ T'.D_x y \prec z \prec D_x(y\tilde{+}1_{OT}) \rightarrow z \not\eta\ G'$.

Proof: (a), (b): Assume $z\ \eta\ G'$. If $z =_N D_I d$ follows $x^- \prec z \prec z$, which yields a contradiction.
If $z =_N D_d e$ with $d \neq_N I$ follows in (a) $x^- \prec d \preceq x$, in (b) $x \preceq d \preceq x$, in both cases therefore $d =_N x$, further in (a) $e \prec 0_{OT}$ and in (b) $y \prec e \prec y\tilde{+}1_{OT}$, both a contradiction.
If $z =_N \widehat{\Omega}_d$ follows in (a) $x^- \prec z \prec x$, in (b) $x \preceq z \prec x$, both yields a contradiction.

Here follow some properties of $Dicom(a)$:

Lemma 9.20 *(a) Let* $a\ \eta\ T'$, $\zeta_0 := Dicom(a)$, $\zeta_{Sn} := \widehat{\Omega}_{\zeta_n}$.
Then $Dicom(a) =_N 0_{OT}$ *and* $a \prec \zeta_n \prec D_I 0_{OT}$ *for some* $n : N$, *or*
$Dicom(a) =_N D_I b$ *for some* b *and* $D_I b \preceq a \prec \zeta_n \prec D_I(b\tilde{+}1_{OT})$ *for some* $n : N$,
or $Dicom(a) =_N I$ *and* $I \preceq a$,
especially we habe $\forall a\ \eta\ T'.Dicom(a) \preceq a$.

(b) If $\pi\ \eta\ R'$ *and* $a\ \eta\ T'$, *then* $G_\pi Dicom(a) \subset G_\pi a$.

Proof:
(a) By induction on $length(a)$: If $a =_N 0_{OT}, b\tilde{+}c, \widehat{\phi}_b c, D_I c, \widehat{\Omega}_b, I$ this follows trivially or by IH, if $a =_N D_b c$ with $I \prec b$ follows $Dicom(a) =_N I \prec a$, and if $a =_N D_b c$ with $b \prec I$ follows $Dicom(a) =_N D_I d$ with $D_I d \preceq b \prec \zeta_n \prec D_I(d\tilde{+}1_{OT})$, $D_I d \prec b$ since $b\ \eta\ R'$, and $D_I d \prec D_b c \prec \zeta_n \prec D_I(d\tilde{+}1_{OT})$.
(b) By induction on $length(a)$: If $a =_N 0_{OT}, b\tilde{+}c, \widehat{\phi}_b c, D_I c, \widehat{\Omega}_b, I$ follows the assertion trivially or by IH, if $a =_N D_b c$ with $I \prec b$ follows $G_\pi Dicom(a) =_{\mathcal{P}^{fin}(N)} G_\pi I =_{\mathcal{P}^{fin}(N)} \emptyset$ and if $a =_N D_b c$ with $b \prec I$ follows, if $\pi \preceq Dicom(a) \preceq a$ or $\pi =_N I$, $G_\pi Dicom(a) \subset G_\pi b \subset G_\pi D_b c$ by IH, and if $Dicom(a) \prec \pi \neq_N I$ follows $G_\pi(Dicom(a)) =_{\mathcal{P}^{fin}(N)} \emptyset$.

Now we are ready to define the fundamental sequences in OT. We will start with a version, where $a[[z]] =_N z$ for $a\ \eta\ R''$, where the ordinals only behave well as long as $\tau(a)^- \prec z$ (in fact they behave well as long as $Dicom(a) \prec z$, the problem occurs only when we have fixed points of $\Omega.$, but the approach starting with $\tau(a)^-$ seems to be a little bit more uniform). In the well-ordering proof we use $a[z] := a[[\tau(a)^- + z]]$. For the analysis of the sequences, it seems to be easier to introduce and analyze first $\cdot[[\cdot]]$, where we do not have problems with the sum and in a second step to define later $\cdot[\cdot]$ and transfer the properties proven.

Definition 9.21 *Definition of* $\tau(a)$ *und* $a[[\xi]]$ *for* $a\ \eta\ T''$, $\xi\ \eta\ T''$, $\xi \prec \tau(a)$, *such that we can define it as primitive recursive functions, have*
$\tau : N \rightarrow N\ \cdot[[\cdot]] : N \rightarrow N \rightarrow N$, *and we have*
$\forall x\ \eta\ T''.\tau(x)\ \eta\ \{0_{OT}, 1_{OT}, \omega\} \cup R''$ *and* $\forall x\ \eta\ T''.\forall \xi\ \eta\ T''.\xi \prec \tau(x) \rightarrow x[[\xi]]\ \eta\ T''$.
If $\tau(a) =_N \omega$ *we define only* $a[[1_{OT} \cdot n]]$, $a[[x]] := a[[1_{OT} \cdot length(x)]]$ *for arbitrary* $x \prec \omega$ *which is for* $x =_N 1_{OT} \cdot n$ *consistent with the first definition.*
For $x \not\eta\ T''$ $\tau(x) := 0_{OT}$, *and for* $x \not\eta\ T'' \vee \xi \not\eta\ T'' \vee \neg(\xi \prec \tau(x))$ $x[[\xi]] := 0_{OT}$.

([[ ]].0) $\tau(0_{OT}) := \emptyset$.

([[ ]].1) *If* $a =_N b\tilde{+}c$, $b\ \eta\ T''$, $c\ \eta\ A''$, *then*
$\tau(a) := \tau(c)$, $(b\tilde{+}c)[[\xi]] := b\hat{+}(c[[\xi]])$

([[ ]].2) *Case* $a =_N \widehat{\phi}_b c$:

([[ ]].2.1) *Case* $b =_N 0_{OT}$:

([[ ]].2.1.1) $c =_N 0_{OT} \Rightarrow \tau(a) := 1_{OT},\ a[[\xi]] := 0_{OT}$

([[ ]].2.1.2) $c =_N c' \tilde{+} 1_{OT} \Rightarrow \tau(a) := \omega,$
$a[[n]] := \widehat{\phi}_{0_{OT}}(c') \cdot SSn.$

([[ ]].2.1.3) $(c\ \eta\ Cr(b)) \Rightarrow \tau(a) := \omega,$
$a[[n]] := c \cdot Sn.$

([[ ]].2.1.4) $(c\ \eta\ Lim'' \wedge c\ \not\eta\ Cr(b)) \Rightarrow \tau(a) := \tau(c),$
$a[[\xi]] := \widehat{\phi}_{0_{OT}}(c[[\xi]])$

([[ ]].2.2) *Case* $b =_N b' \tilde{+} 1_{OT}$:

([[ ]].2.2.1) $c =_N 0_{OT} \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_{SSSn}$ *where* $\rho_0 := 0_{OT},\ \rho_{Sn} := \widehat{\phi}_{b'}\rho_n$

([[ ]].2.2.2) $c =_N c' \tilde{+} 1_{OT} \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_n$ *where* $\rho_0 := \widehat{\phi}_b c',\ \rho_{Sn} := \widehat{\phi}_{b'}\rho_n$

([[ ]].2.2.3) $(c\ \eta\ Cr(b)) \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_{Sn}$, *where* $\rho_0 := c,\ \rho_{Sn} := \widehat{\phi}_{b'}\rho_n$

([[ ]].2.2.4) $(c\ \eta\ Lim'' \wedge (c\ \not\eta\ Cr(b)) \Rightarrow \tau(a) := \tau(c),$
$a[[\xi]] := \widehat{\phi}_b(c[[\xi]])$

([[ ]].2.3) *Case* $b\ \eta\ Lim''$:

([[ ]].2.3.1) $(c =_N 0_{OT} \wedge b\ \eta\ T'' \setminus G'') \Rightarrow \tau(a) := \tau(b),$
$a[[\xi]] := \widehat{\phi}_{b[[\xi]]} 0_{OT}$

([[ ]].2.3.2) $(c =_N 0_{OT} \wedge b\ \eta\ G'') \Rightarrow \tau(a) := \tau(b),$
$a[[\xi]] := \widehat{\phi}_{b[[\xi]]} b$

([[ ]].2.3.3) $c =_N c' \tilde{+} 1_{OT} \Rightarrow \tau(a) := \tau(b),$
$a[[\xi]] := \widehat{\phi}_{b[[\xi]]}(\widehat{\phi}_b c')$

([[ ]].2.3.4) $(c\ \eta\ Cr(b)) \Rightarrow \tau(a) := \tau(b),$
$a[[\xi]] := \widehat{\phi}_{b[[\xi]]} c$

([[ ]].2.3.5) $(c\ \eta\ Lim'' \wedge c\ \not\eta\ Cr(b)) \Rightarrow \tau(a) := \tau(c),$
$a[[\xi]] := \widehat{\phi}_b(c[[\xi]])$

([[ ]].3) *Case* $a =_N D_b c$:

([[ ]].3.1) *Case* $c =_N 0_{OT}$:

([[ ]].3.1.1) $(b \neq_N I \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_{Sn}$, *where* $\rho_0 := b^-,\ \rho_{Sn} := \widehat{\phi}_{\rho_n} 0_{OT}.$

([[ ]].3.1.2) $b =_N I \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_{SSn}$, *where* $\rho_0 := 0_{OT},\ \rho_{Sn} := \widehat{\Omega}_{\rho_n}.$

([[ ]].3.2) *Case* $c =_N c' \tilde{+} 1_{OT}$:

([[ ]].3.2.1) $(b \neq_N I \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_n$, *where* $\rho_0 := D_b c',\ \rho_{Sn} := \widehat{\phi}_{\rho_n} 0_{OT}.$

([[ ]].3.2.2) $b =_N I \Rightarrow \tau(a) := \omega,$
$a[[1_{OT} \cdot n]] := \rho_n$, *where* $\rho_0 := D_I c',\ \rho_{Sn} := \widehat{\Omega}_{\rho_n}.$

([[ ]].3.3) *Case* $c\ \eta\ Lim'',\ \tau(c) \prec b$:
$\tau(a) := \tau(c),\ a[[\xi]] := D_b c[[\xi]].$

([[ ]].3.4) *Case* $c\ \eta\ Lim'',\ b \preceq \tau(c)$:
Then $\tau(a) := \omega,\ a[[n]] := D_b c[[\zeta_n]]$, *where* ζ_n *is defined by:*

([[ ]].3.4.1) $(\tau(c) \neq_N I \vee b \prec D_I c \vee b =_N I \Rightarrow$
$\zeta_0 := \pi^-,\ \zeta_{Sn} := D_\pi(c[[\zeta_n]]),\ \pi := \tau(c).$

([[ ]].3.4.2) $(\tau(c) =_N I \wedge D_I c \preceq b \prec I) \Rightarrow$
$\zeta_0 := Dicom(b),\ \zeta_{Sn} := \widehat{\Omega}_{\zeta_n}.$

([[ ]].4) *Case* $a =_N \widehat{\Omega}_b$:
([[ ]].4.1) *Case* $a\ \eta\ R''$:
$\tau(a) := a,\ a[[\xi]] := \xi$
([[ ]].4.2) *Case* $a\ \not\eta\ R''$:
$\tau(a) := \tau(b),\ a[[\xi]] := \widehat{\Omega}_{b[[\xi]]}$
([[ ]].5) *Case* $a =_N I$:
$\tau(a) := I,\ a[[\xi]] := \xi$

We define $\widehat{\tau(:)} = \lambda y.\lambda\xi.\xi\ \eta\ T'' \wedge \xi \prec \tau(y) \wedge \tau(y)^- \preceq \xi$, *and have* $\widehat{\tau(:)}N \to \mathcal{P}^{dec}(N)$, $\forall x\ \eta\ T''.\widehat{\tau(x)} \subset T''$.

Lemma 9.22 *(a)* $\forall x,\xi\ \eta\ T'.\xi \prec \tau(x) \to x[[\xi]]\ \eta\ T' \wedge x[[\xi]] \prec x \wedge \tau(x) \preceq x$.

(b) In the situation of [[ ]].3.4 *we have* $\forall n \in N.\zeta_n \prec \tau(c)$, *and if* $b, c \in T'$ *then* $\forall n \in N.\zeta_n \in T'$.

(c) $\forall x, y\ \eta\ T'.y \neq_N 0_{OT} \to NF_+(x,y) \to$
$(\tau(x\hat{+}y) =_N \tau(y) \wedge \forall\xi\ \eta\ \widehat{\tau(y)}.(x\hat{+}y)[[\xi]] =_N x\hat{+}(y[[\xi]]))$.

Proof: (a), (b): (a) follows by an easy induction on $length(x)$, using in ([[]].1) $c[[\xi]] \prec c \to a\tilde{+}c\ \eta\ T' \to a\hat{+}c[[\xi]]\ \eta\ T'$ by 9.13 (h) and proving in the situation of [[ ]].3.4 (b) by side induction on N.
(c) Easy induction on $Alength(y)$.

Lemma 9.23 *Assume* $a, \xi, \rho\ \eta\ T'$.

(a) $\xi \prec \rho \prec \tau(a) \to a[[\xi]] \prec a[[\rho]]$.

(b) $(\omega \prec \tau(a) \wedge \xi \preceq \tau(a)) \to (\xi \preceq a[[\xi]] \wedge \forall\pi\ \eta\ R'.(\pi \neq I \vee \pi \preceq \tau(a)) \to G_\pi\xi \preceq G_\pi a[[\xi]])$ *(where* $a[[\tau(a)]] := a$*).*

(c) $(0 \neq a \wedge a\ \not\eta\ R') \leftrightarrow \tau(a) \prec a$.

(d) $(a\ \eta\ OT \to \tau(a)\ \eta\ OT) \wedge$.

(e) $a\ \not\eta\ G' \to a\ \eta\ Lim' \to 1 \prec x[[\tau(a)^-]]$.

Proof:
(a) By an induction on $length(a)$ in an easy way, where in the cases $\tau(a) = \omega$ we show first by induction $n := length(\xi)\ length(\rho) = Sn \to a[[\xi]] \prec a[[\rho]]$ and the assertion follows by side induction on $length(\rho) - length(\xi)$.
(b) Case $a = b\tilde{+}c$: $\xi \preceq c[[\xi]] \preceq b\hat{+}c[[\xi]]$
Case $a = \widehat{\phi}_b c$: If $c\ \eta\ Lim' \setminus Cr(b)$ follows $\xi \preceq c[[\xi]] \preceq \widehat{\phi}_b c[[\xi]]$ by IH and $G_\pi(\xi) \preceq G_\pi c[[\xi]] \preceq G_\pi a[[\xi]]$. If $b\ \eta\ Lim' \wedge ((c\ \not\eta\ Lim' \vee c\ \eta\ Cr(b))$ follows $\xi \preceq b[[\xi]] \preceq \widehat{\phi}_{b[[\xi]]}e = a[[\xi]]$ for some e and $G_\pi(\xi) \preceq G_\pi b[[\xi]] \preceq G_\pi a[[\xi]]$.
If $a = D_b c\ c\ \eta\ Lim'$, $\tau(c) \prec b$ follows $\xi \prec D_b(\xi) \preceq D_b c[[\xi]]$ by lemma 9.15 (h) and $\xi \preceq c[[\xi]]$. Further, if $\pi \preceq b \neq I$ or $b = I andd \pi \prec D_I c[[\xi]]$ follows $G_\pi\xi \preceq G_\pi c[[\xi]] \preceq G_\pi(D_b c[[\xi]])$. If $b \prec \pi \neq I \vee (I = b \wedge D_I c[[\xi]] \prec \pi \prec I)$ follows, since $\xi \preceq \tau(c) \prec b \prec \pi \neq I$ or $\xi \prec D_I\xi \preceq D_I c[[\xi]] \prec \pi \neq I\ G_\pi(\xi) \cong \emptyset$. $b \preceq \pi = I$ is not possible by assumption.
If $a = \widehat{\Omega}_b$, and if $a[[\xi]] = a^- + \xi$, follows the assertion immediately and if $a[[\xi]] = \widehat{\Omega}_{b[[\xi]]}$ follows $\xi \preceq b[[\xi]] \prec \widehat{\Omega}_{b[[\xi]]}$, $G_\pi(\xi) \preceq G_\pi b[[\xi]] \cong G_\pi a[[\xi]]$.
(c) Trivial.
(d): Induction on $length(c)$: The cases $\tau(a) = 0, 1, \omega$ or $\tau(a) = a$ are trivial, otherwise the assertion follows by IH.

(e) If $x = b\widetilde{+}c$, then $\omega \preceq c \preceq b \preceq x[[\tau(x)^-]]$.
If $x = \widehat{\phi}_b c$, follows, if $b = 0$, $c = c'\widetilde{+}1$, $1 \prec \widehat{\phi}_0 c' \cdot SSn$, if $c \,\eta\, Lim' \setminus Cr(b) \wedge b = 0$ follows $\neg(c \,\eta\, G)$, $1 \prec c[[\tau(c)^-]]$, $1 \prec x[[\tau(x)^-]]$, and if $c \,\eta\, Lim' \setminus Cr(b) \wedge 0 \prec b$, $1 \prec \widehat{\phi}_b c[[\tau(b)^-]] = x[[\tau(x)^-]]$.
If $b = 0$, $c \,\eta\, G'$ follows $1 \prec c \cdot Sn$.
If $b = b'\widetilde{+}1$, $c \,\not\eta\, Lim' \setminus C(b)$, follows $x[[n]] = \widehat{\phi}_{b'} d$ for some $d \neq 0$ or $x[[n]] = \widehat{\phi}_b c'$ and we have the assertion.
If $b \,\eta\, Lim' \setminus G'$, $c = 0$ follows $1 \prec b[[\tau(b)^-]] \prec x[[\tau(x)^-]]$.
If $b \,\eta\, G'$, $c = 0$ follows $1 \prec \widehat{\phi}_{b[[\tau(b)^-]]} b$.
If $\beta \,\eta\, Lim'$, $c = c'\widetilde{+}1$, follows $1 \prec \widehat{\phi}_{b[[\tau(b)^-]]} \widehat{\phi}_b c'$, since $0 \prec \widehat{\phi}_b c'$.
If $c \,\eta\, Cr(b)$, $b \,\eta\, Lim'$, follows $1 \prec \widehat{\phi}_{b[[\tau(b)^-]]} c = x[[\tau(x)^-]]$.

We will now introduce a^* for $a \,\eta\, T'$, which will be some sort of predecessor of a. a^* has the property, that $lh'(a^*) <_N lh'(a)$, so, we have $\forall a \,\eta\, T'.\exists n \in N.s\overbrace{^{*\cdots *}}^{}{}_{ntimes} = 0$, and we can use induction over the length of the descending sequences. We will use $\cdot^*$ to prove the property $a[[x]] \prec b \preceq a[[x\widetilde{+}1]] \rightarrow a[[x]] \preceq b[[\tau(b)^-]]$.

Definition 9.24 *(a) Definition of* $\cdot^* : N \rightarrow N$ *such that* $s^* \,\eta\, T'$ *for* $s \,\eta\, T'$, $s^* \,\eta\, T''$ *for* $s \,\eta\, T''$ *(*$s^* := 0$ *for* $s \,\not\eta\, T''$*).*

$0^* := I^* := 0$.

$(a\widetilde{+}b)^* := a\widehat{+}(b^*)$.

$$(\widehat{\phi}_a b)^* := \begin{cases} \widehat{\phi}_a(b^*) & \text{if } b \,\not\eta\, Cr(a) \cup \{0\}, \\ b & \text{if } a = 0 \wedge b \,\eta\, Cr(a) \cup \{0\} \\ a & \text{if } b = 0 \wedge a \,\eta\, G' \\ \widehat{\phi}_0 0 & \text{if } b = 0 \wedge a = 1 \\ \widehat{\phi}_{a^*} b & \text{if } (b \,\eta\, Cr(a) \wedge a \neq 0) \vee (b = 0 \wedge \\ & \quad a \,\not\eta\, G' \cup \{0,1\}) \end{cases}.$$

$\widehat{\Omega}^*_0 := \omega$, $\widehat{\Omega}^*_a := a$ *for* $a \,\eta\, Fi'$, $\widehat{\Omega}^*_a := \widehat{\Omega}_{a^*}$ *for* $a \,\not\eta\, Fi' \cup \{0\}$

$$(D_\pi b)^* := \begin{cases} D_\pi(b^*) & \text{if } b \neq 0 \\ \pi^- & \text{if } b = 0 \wedge \pi \neq I.. \\ \widehat{\Omega}_0 & \text{if } b = 0 \wedge \pi = I \end{cases}$$

(b) Definition of $lh' : N \rightarrow N$. $lh'(n) := 0$ *for* $n \,\not\eta\, T'$.
$lh'(0) := 0$, $lh'(I) := 1$, $lh'(a\widetilde{+}b) := lh'(a) +_N lh'(b) +_N 1$, $lh'(\widehat{\phi}_a b) := lh'(a) +_N lh'(a) +_N lh'(b) +_N 1$, $lh'(D_a b) := lh'(a) +_N lh'(b) +_N 3$, $lh'(\widehat{\Omega}_a) := lh'(a) +_N 3$.

Lemma 9.25 *Let* $a, b, c \,\eta\, T'$.

(a) $(c\widetilde{+}1)^* = c$, $\omega^* = 1$, $\widehat{\phi}_a(b\widetilde{+}1)^* = \widehat{\phi}_a b$.

(b) $c \neq 0 \rightarrow lh'(c^*) <_N lh'(c)$.

(c) If $n : N$, *follows* $(1 \cdot (Sn))^* = 1 \cdot n$.

(d) If $b \neq 0$, *then* $(a\widehat{+}b)^* = a\widehat{+}(b^*)$.

(e) $a \neq 0 \rightarrow a^* \prec a$.

(f) $b^* \preceq (\widehat{\phi}_a b)^*$.

Proof:
(a) By definition.
(b) Easy induction on $length(c)$, using that for $a \,\eta\, R'$ $a^- = a^*$.
(c) By induction on $n' : N$.
(d) By induction on $Alength(b)$.
(e) Immediate by induction on $length(a)$.
(f) Immediate, using $b^* \preceq b$.

Lemma 9.26 $s \,\eta\, T' \rightarrow s^* \preceq s[\![\tau(s)^-]\!]$.

Proof:
Induction on $length(s)$.
If $s = 0$, the assertion is trivial.
If $s \,\eta\, R'$ follows $s^* = s[\![\tau(s)^-]\!]$.
If $s = a\widetilde{+}b$ follows $b^* \preceq b[\![\tau(b)^-]\!]$, $a\widehat{+}(b^*) \preceq a\widehat{+}b[\![\tau(b)^-]\!]$.
Case $s = \widehat{\phi}_a b$: If $(a = 0 \vee a = a'\widetilde{+}1)$ and $(b = 0 \vee b = b'\widehat{+}1 \vee b \,\eta\, Cr(a))$ follows immediately $s^* \preceq s[\![\tau(s)^-]\!]$.
If $b \,\eta\, Lim' \setminus Cr(a)$ follows $b^* \preceq b[\![\tau(b)^-]\!]$ by IH therefore $\widehat{\phi}_a(b^*) \preceq \widehat{\phi}_a(b[\![\tau(b)^-]\!]$.
If $a \,\eta\, Lim' \setminus G'$, $b = 0$ follows $s^* = \widehat{\phi}_{a^*}0 \preceq \widehat{\phi}_{a[\![\tau(a)^-]\!]}0 = s[\![\tau(s)^-]\!]$ since $a^* \preceq a[\![\tau(a)^-]\!]$ by IH and $0 \preceq \widehat{\phi}_{a[\![\tau(a)^-]\!]}0$.
If $a \,\eta\, G'$, $b = 0$ follows $s^* = a \preceq \widehat{\phi}_{a[\![\tau(a)^-]\!]}a = s[\![\tau(s)^-]\!]$.
If $a \,\eta\, Lim'$, $b = b'\widetilde{+}1$ follows $s^* = \widehat{\phi}_a b' \preceq \widehat{\phi}_{a[\![\tau(a)^-]\!]}(\widehat{\phi}_a b') = s[\![\tau(s)^-]\!]$.
If $a \,\eta\, Lim'$, $b \,\eta\, Cr(a)$, follows $s^* = \widehat{\phi}_{a^*}b \preceq \widehat{\phi}_{a[\![\tau(a)^-]\!]}b = s[\![\tau(s)^-]\!]$.
Case $s = D_a b$:
If $b \,\not\eta\, Lim'$ follows $s[\![\tau(s)^-]\!] \preceq s^*$ immediately.
If $b \,\eta\, Lim'$, $\tau(b) \prec a$ or $a \preceq \tau(b) \neq I$ or $a \preceq \tau(b) = I \wedge (a \prec D_I b \vee a = I)$ follows $s^* = D_a(b^*) \preceq D_a(b[\![\tau(b)^-]\!]) = s[\![\tau(s)^-]\!]$.
If $b \,\eta\, Lim'$, $a \preceq \tau(b) = I$, $D_I b \preceq a \prec I$ follows $s^* = D_a b^* \preceq D_a(b[\![\tau(b)^-]\!]) \preceq D_a(b[\![Dicom(b)]\!]) = s[\![\tau(s)^-]\!]$.
Case $s = \widehat{\Omega}_a$, $s \,\not\eta\, R'$: $a^* \preceq a[\![\tau(a)^-]\!]$, $\widehat{\Omega}_a^* \preceq \widehat{\Omega}_{a[\![\tau(a)^-]\!]}$.

Lemma 9.27 *Assume $a, b, c \,\eta\, T''$.*

(a) $(a \,\not\eta\, Cr(c) \wedge a^* \prec b \preceq a) \rightarrow b \,\not\eta\, Cr(c)$.

(b) $(a \,\not\eta\, G' \wedge a^* \prec b \preceq a) \rightarrow b \,\not\eta\, G'$.

(c) $(a \,\not\eta\, Fi' \cup R' \wedge a^* \prec b \preceq a) \rightarrow b \,\not\eta\, Fi' \cup R'$.

(d) $(a \,\not\eta\, Fi' \wedge a^* \prec b \preceq a) \rightarrow b \,\not\eta\, Fi'$.

(e) $b \,\eta\, Cr(a) \cup \{0\} \rightarrow (b \prec c \preceq \widehat{\phi}_a b \rightarrow b \preceq c^*)$

(f) $b \,\eta\, \{1, \omega, \widehat{\Omega}_0, \} \rightarrow (b \prec a \prec I \rightarrow b \preceq a^*)$.
$\forall a \,\eta\, T'.I \prec a \rightarrow I \preceq a^*$.

Proof:
(a) Induction on $length(a)$, side induction on $length(b)$. Assume $a, b, c \,\eta\, T'$, such that $a \,\not\eta\, Cr(c)$ and $a^* \prec b \preceq a$, $b \,\eta\, Cr(c)$
Case $a = d\widetilde{+}e$: Then $b = d\widehat{+}f$ such that $e^* \prec f \preceq e$, especially $f \neq 0$.
Case $a = \widehat{\phi}_d e$: Then $d \preceq c$.

Case $e \not\eta\, Cr(d) \cup \{0\}$: Then $\widehat{\phi}_d e^* \prec b \preceq \widehat{\phi}_d e$, and since $b\ \eta\ Cr(c) \subset Cr(d)$, $e^* \prec b \preceq e$, and, since $e \not\eta\, Cr(d)$, $e \not\eta\, Cr(c)$, $b \not\eta\, Cr(c)$.

Case $d = 0$ and $e\ \eta\ Cr(d) \cup \{0\}$. Then $e \prec b \preceq \widehat{\phi}_d e$, and by lemma 9.18 (c) and (e) follows $b \not\eta\, Cr(d)$, $b \not\eta\, Cr(a)$.

Case $e = 0 \wedge d\ \eta\ G'$: Then by lemma 9.18 (c) follows the assertion.

Case $(e\ \eta\ Cr(d) \wedge d \neq 0) \vee (e = 0 \wedge d\ \eta\ T' \setminus (G' \cup \{0,1\})$. Then $\widehat{\phi}_{d^*} e \prec b \preceq \widehat{\phi}_d e$, and since $b\ \eta\ Cr(c) \subset Cr(d)$, $e \prec b \preceq e$, a contradiction.

Case $d = 1 \wedge e = 0$: 9.18 (c).

Case $a\ \eta\ G'$: $b \preceq a \preceq c$, $b \not\eta\, Cr(c)$.

(b) Induction on $length(a)$, side induction on $length(b)$. Assume $a, b\ \eta\ T'$, such that $a \not\eta\, G'$ and $a^* \prec b \preceq a$, $b\ \eta\ G'$

Case $a = d \tilde{+} e$: Then $b = d \tilde{+} f$ such that $e^* \prec f \preceq e$, especially $f \neq 0$.

Case $a = \widehat{\phi}_d e$:

If $e \not\eta\, Cr(d) \cup \{0\}$ or $d = 0$ or $e = 0 \wedge d\ \eta\ G'$ follows $d \preceq (\widehat{\phi}_d e)^* \prec b$, and if we had $b\ \eta\ G'$ we had $b\ \eta\ Cr(d)$ contradicting (a). If $e\ \eta\ Cr(d) \wedge d \neq 0$ follows $d \preceq e \preceq \widehat{\phi}_{d^*} e \prec b$ and by the same argument the assertion, and if $e = 0 \wedge d \not\eta\, G' \cup \{0,1\}$ follows from $b\ \eta\ G'$, $d^* = max_T\{d^*, e\} \prec b \preceq max_T\{d, e\} = d$, contradicting $d \not\eta\, G'$ and the IH. If $d = 1 \wedge e = 0$ follows $b \prec \widehat{\phi}_1 0 \prec d_{\widehat{\Omega}_0} 0$, $b \not\eta\, G'$.

(c) Induction on $length(a)$, side induction on $length(b)$. Assume $a, b\ \eta\ T'$, such that $a \not\eta\, Fi' \cup R'$ and $a^* \prec b \preceq a$, $b\ \eta\ G'$

Case $a = 0$, $a = d \tilde{+} e$, $\widehat{\phi}_d e$: Then $b \not\eta\, G'$ by (b).

Case $a = D_d e$, $d \neq I$: Then $d^- \preceq a^* \prec b \prec d$ which is not possible.

(d) Induction on $length(a)$, side induction on $length(b)$. Assume $a, b\ \eta\ T'$, such that $a \not\eta\, Fi'$ and $a^* \prec b \preceq a$, $b\ \eta\ Fi'$

Case $a = 0$, $a = d \tilde{+} e$, $\widehat{\phi}_d e$: Then $b \not\eta\, G'$ by (b).

Case $a = D_d e$, $d \neq I$: Then $b \not\eta\, Fi'$ by (c).

Case $a = \widehat{\Omega}_d$. If $d\ \eta\ Fi'$ follows $d = a^* \prec b \preceq a$, $d \prec b \preceq d$, not possible, and if $d \not\eta\, Fi' \cup \{0\}$ follows $a^* = \widehat{\Omega}_{d^*}$, $d^* \prec b \preceq d$ and the assertion by IH. If $d = 0$ follows $a \prec D_I 0$, $a \not\eta\, Fi'$.

(e) The case $b = 0$ is trivial. Let $b\ \eta\ Cr(a)$. Induction on $length(c)$: By lemma 9.18 (e) $c \not\eta\, Cr(a) \cup G'$, therefore $c = d \tilde{+} e$ or $c = \widehat{\phi}_d e$ with $d \preceq a$.

If $c = d \tilde{+} e$ follows $b \preceq d \preceq c^*$.

Case $c = \widehat{\phi}_d e$.

Subcase $d \prec a$: Then $b \preceq e \preceq \widehat{\phi}_a b$. If $b = e\ \eta\ Cr(d)$ follows $c^* = e$ or $c^* = \widehat{\phi}_{d^*} e$, $b \preceq c^*$. If $b \prec e$ follows by IH $b \preceq e^*$, $c^* = \widehat{\phi}_d e^*$ or $c^* = e$ or $c^* = \widehat{\phi}_{d^*} e$, $b \preceq c^*$.

Subcase $d = a$: Then $b \preceq e \preceq b$, $c = \widehat{\phi}_a b$, $c^* = b$ or $c^* = \widehat{\phi}_{a^*} b$ or $c^* = \widehat{\phi}_0 1 \wedge d = 1 \wedge e = 0 = b$.

(f) Immediate.

Lemma 9.28 $\forall s, t\ \eta\ T'.s^* \prec t \preceq s \rightarrow s^* \preceq t^*$.

Proof: Induction on $length(s) +_N length(t)$.

Case $s = a \tilde{+} b$: $a \widehat{+} b^* \prec t \prec a \tilde{+} b$, therefore $t = a \widehat{+} c$ for some $b^* \prec c \preceq b$, $b^* \preceq c^*$, $s^* = a \widehat{+} b^* \preceq t^*$.

Case $s = \widehat{\phi}_a b$: Then by lemma 9.27 (a), (b) $t \not\eta\, Cr(a) \cup G'$, therefore $t = a \tilde{+} b$ (in which case, since $s^*\ \eta\ A$, $s^* \preceq a \preceq t^*$), or $t = \widehat{\phi}_c d$ with $c \preceq a$. Let $t = \widehat{\phi}_c d$.

Subcase $b \neq 0$, $b\ \eta\ T' \setminus Cr(a)$: $\widehat{\phi}_a b^* \prec t \preceq \widehat{\phi}_a b$.

If $c \prec a$ $\widehat{\phi}_a b^* \preceq d \preceq \widehat{\phi}_a b$. If $d = s^*$, follows $d\ \eta\ Cr(c) \setminus G'$, $t^* = s^*$ or $t^* = \widehat{\phi}_{c^*} d$, therefore $s^* \preceq t^*$. If $s^* \prec d$ follows by IH $s^* \preceq d^*$, $d \neq 0$, therefore $t^*\ \eta\ \{d, \widehat{\phi}_c d^*, \widehat{\phi}_{c^*} d\}$, $s^* \preceq t^*$. If $c = a$, follows $b^* \prec d \preceq b$, by IH $b^* \preceq d^*$, and, since $b \not\eta\, Cr(a)$, $d \not\eta\, Cr(a)$ and $d \neq 0$. Therefore $s^* \preceq \widehat{\phi}_a b^* = t^*$.

Subcase $a = 0 \wedge b\ \eta\ Cr(a) \cup \{0\}$: by lemma 9.27 (e).

Subcase $b = 0 \wedge a \,\eta\, G'$. If $c \prec a$ follows $a \,\eta\, Cr(c)$, $a \preceq d \preceq \widehat{\phi}_a b$. If $d = a$ follows $t^* = d$ or $t^* = \widehat{\phi}_{c^*} d$, $a \preceq t^*$, and if $a \prec d$, $a \preceq d^* \preceq t^*$. If $c = a$ follows $d \preceq b = 0$, $s = t$.
Subcase $(b \,\eta\, Cr(a) \wedge a \neq 0) \vee (b = 0 \wedge a \,\eta\, G' \setminus \{0,1\})$:
Then $\widehat{\phi}_{a^*} b \prec t \preceq \widehat{\phi}_a b$.
Subsubcase $c \prec a^*$: Then $s^* \preceq d \preceq s$. If in this case $d = s^*$, follows $t^* = d$ or $t^* = \widehat{\phi}_{c^*} d$, $s^* \preceq t^*$, and if $s^* \prec d$ follows $s^* \preceq d^* \preceq t^*$.
Subsubcase $c = a^*$: $b \prec d \preceq \widehat{\phi}_a b$, by lemma 9.27 (e) $b \preceq d^*$. If then $t^* = \widehat{\phi}_c d^*$, follows $s^* \preceq t^*$. If $t^* = d$, $d \,\eta\, Cr(c) \cup \{0\}$ follows $d \,\eta\, Cr(a^*)$, by $b \prec d$, $\widehat{\phi}_{a^*} b \preceq d$. The case $t^* = c$ does not occur, because $d \neq 0$. The case $t^* = \widehat{\phi}_{c^*} d$ occurs only, in case of $d \,\eta\, Cr(c)$, $c \neq 0$, and by $b \prec d$, follows $\widehat{\phi}_{a^*} b \prec d \preceq \widehat{\phi}_{c^* d}$. The case $t = \widehat{\phi}_1 0$ is not possible.
Subsubcase $a^* \prec c \preceq a$: If $b = 0 \wedge a \,\eta\, G' \setminus \{0,1\}$ follows $a^* \preceq c^*$, $s^* = \widehat{\phi}_{a^*} 0 \preceq \widehat{\phi}_c d^* = t^*$ or $s^* = \widehat{\phi}_{a^*} 0 \preceq c = t^*$ where $c \,\eta\, G'$, or $s^* \preceq \widehat{\phi}_{c^*} d \preceq t^*$. If $b \,\eta\, Cr(a) \wedge a \neq 0$ follows from $s^* \prec t \preceq s$, $b \prec \widehat{\phi}_c d \preceq \widehat{\phi}_a b$, $b \preceq d \preceq \widehat{\phi}_a b$, $b \preceq d^* \preceq t^*$ or $b = d \,\eta\, Cr(c)$, $s^* = \widehat{\phi}_{a^*} 0 \preceq \widehat{\phi}_{c^*} d = t^*$.
Subcase $a = 1 \wedge b = 0$: $\omega \prec s \prec I$, by 9.27 (f) follows $\omega \preceq s^*$.
Case $s = D_a b$: $s^* \,\eta\, G'$, so in case of $t \not\eta\, G'$ follows by lemma 9.27 (b) $s^* \preceq t^*$. Let therefore $t \,\eta\, G'$.
Subcase $b \neq 0$: Then $D_a(b^*) \prec t \preceq D_a b$.
Subsubcase $a \neq I$: Then $t = D_a c$ with $b^* \prec c \preceq b$, $b^* \preceq c^*$, $s^* \preceq D_a c^* = t^*$.
Subsubcase $a = I$: If $t = D_c d$ with $c \neq I$ follows $D_a b^* \preceq c^- \preceq t^*$. If $c = I$ follows the assertion as in the case $b \neq I$. If $t = \widehat{\Omega}_c$ follows $s^* \preceq c \preceq s$, if $c = s^* = D_I b^*$ follows $t^* = s^*$, if $s^* \prec c$, $s^* \preceq c^* \preceq t^*$.
Subcase $b = 0 \wedge a \neq I$: $a^- \prec t \prec a$, $t = D_a d$, $d \preceq 0$, therefore $t = s$.
Subcase $b = 0 \wedge a = I$: Then $\widehat{\Omega}_0 \prec t \preceq D_I 0$, and by lemma 9.27 (f) $\widehat{\Omega}_0 \preceq t^*$. Case $s = \widehat{\Omega}_a$: If $a = 0$ follows the assertion by lemma 9.27 (f). Otherwise follows $s \,\eta\, G'$ and the assertion in case $t \not\eta\, G'$. Let $t \,\eta\, G'$.
Subcase $s^* = s^-$. Then $t = D_s c$, $s^- \preceq t^*$.
Subcase $s \not\eta\, R'$, $s^* = \widehat{\Omega}_{a^*}$. If $t = D_c d$, with $c \neq I$ follows $\widehat{\Omega}_{a^*} \prec c$, $\widehat{\Omega}_{a^*} \preceq c^* \preceq t^*$. If $t = D_I c$ follows $a^* \prec t \preceq a$, since $t \,\eta\, Fi'$, $a \,\eta\, Fi'$, contradicting $t^* = \widehat{\Omega}_{a^*}$. If $t = \widehat{\Omega}_c$ follows by IH $a^* \prec c \preceq a$, $a^* \preceq c^*$ and the assertion.
Case $s = I$: Then the assertion is trivial.

Definition 9.29 $t \ll s :\Leftrightarrow \exists l \in Nlist.0 <_N lh(l) \wedge (\forall i <_N lh(l).(l)_i \,\eta\, T') \wedge (l)_0 = t \wedge (l)_{pred_N(lh(l))} = s \wedge \forall i \in N.Si <_N lh(l) \rightarrow (l)_{Si} = {(l)_i}^*$,
that is, informally written,
$t \ll s \Leftrightarrow \exists n \in N, \exists s_0, \ldots, s_{Sn} \,\eta\, T'.s_0 = s \wedge s_{Sn} = t \wedge \forall i \leq n.s_{Si} = s_i^*$,
or even more informal:

$$t \ll s \Leftrightarrow \exists n \in N.0 <_N n \wedge s^{\overbrace{* \cdots *}^{ntimes}} = t.$$

In the following lemmata we will refer to the informal definition, and all assertions can easily be transformed into formal proofs, which can not be read any more.
In the situation of the second definition we define $lh_{\ll}(s,t) := n$.

Lemma 9.30 *Let $r, s, t \,\eta\, T'$.*

(a) *For $r \,\eta\, \{0, I\}$ we have $r \prec s \rightarrow r \ll s$, and for $r \,\eta\, \{1, \omega, \widehat{\Omega}_0\}$ we have $r \prec s \prec I \rightarrow r \ll s$.*

(b) $r \ll r \widehat{+} s$, *and* $r \ll s \rightarrow t \widehat{+} r \ll t \widehat{+} s$.

(c) $r \ll s \wedge (\forall x \,\eta\, T'.r \prec x \preceq s \rightarrow (x \not\eta\, Cr(t)) \rightarrow \widehat{\phi}_t r \ll \widehat{\phi}_t s$.

(d) $(r \ll s \wedge (\forall x \,\eta\, T'.r \prec x \preceq s \to (x \not\eta\, G')) \to \widehat{\phi}_r 0 \ll \widehat{\phi}_s 0$,
$(s \,\eta\, Cr(r) \to s \ll \widehat{\phi}_r s)$,
$(t \,\eta\, Cr(s) \wedge r \ll s) \to \widehat{\phi}_r t \ll \widehat{\phi}_s t$.

(e) *If* $r \,\eta\, G'$, $s \,\eta\, T'$, $s \prec \widehat{\phi}_{r\tilde{+}1} 0$, *then* $r \ll \widehat{\phi}_r s$.

(f) $(r \ll s \wedge \pi \,\eta\, R') \to D_\pi r \ll D_\pi s$.

(g) $r^- \prec s \preceq r$, $r \,\eta\, R'$, *then* $r^- \ll s$.

(h) $(r \ll s \wedge \forall x \,\eta\, T'.r \prec x \preceq s \to c \not\eta\, Fi') \to \widehat{\Omega}_r \ll \widehat{\Omega}_s)$.

Proof:
We use $lh'(s^*) <_N lh'(s)$. (a) By induction on $lh'(s)$ since $r \prec s \to r \preceq s^*$ by lemma 9.27 (a).
(b) by induction on $lh'(s)$ follows the first, by induction on $lh_{\ll}(r, c)$ the second assertion.
(c) Induction on $lh_{\ll}(r, s)$. $(\widehat{\phi}_t s_{Si})^* = \widehat{\phi}_t (s_{Si})^* = \widehat{\phi}_t s_i$ since $s_{Si} \not\eta\, Cr(t) \wedge s_{Si} \neq 0$.
(d) Induction on $lh_{\ll}(r, s)$, $lh'(r)$, $lh_{\ll}(r, s)$.
(e) Induction on $lh'(s)$.
(f), (h) Induction on $lh_{\ll}(r, s)$.
(g) Induction on $lh'(s)$. If $s = r$ follows $s^* = r^-$. Otherwise follows if $r^- = \omega$, by lemma 9.27 (f), if $r = I$ trivially, and if $r^- \,\eta\, (R' \cup Fi')$ and $\forall x \,\eta\, T'.r^- \prec x \prec r \to x \not\eta\, R' \cup Fi'$ by lemma 9.27 (c) $r^- \preceq s^*$.
(h) Induction on $lh_{\ll}(r, s)$.

Lemma 9.31 *If* $s, \xi, \rho \,\eta\, T'$, $\tau(s)^- \preceq \xi \ll \rho \prec \tau(s)$, *then* $s[[\xi]] \ll s[[\rho]]$.

Proof: by Induction on $length(s)$. In case of $\tau(s) = \omega$, it is sufficient to show $\forall n \in N.s[[1 \cdot n]] \ll s[[1 \cdot Sn]]$. If $s[[1 \cdot n]] := \rho_{S^i n}$ for some i in Definition 9.21, we will prove $\rho_n \ll \rho_{Sn}$ by side induction on $n : N$.
If $s = 0$, the premise cannot be fulfilled.
If $s = a\tilde{+}b$, the assertion follows by $b[[\xi]] \ll b[[\rho]]$ and by lemma 9.30.
Case $s = \widehat{\phi}_a b$:
Subcase $a = b = 0$ is not possible.
Subcase $a = 0$, $b = b'\tilde{+}1$: $s[[1 \cdot n]] = \widehat{\phi}_a b' \cdot SSn \ll \widehat{\phi}_a b' \cdot SSSn = s[[1 \cdot (Sn)]]$.
Subcase $b \,\eta\, Lim' \setminus Cr(a)$: $b[[\xi]] \ll b[[\rho]]$, $b^* \preceq b[[\tau(b)^-]] \preceq b[[\xi]] \prec b[[\rho]] \preceq b$, therefore by lemma 9.27 (a) and 9.30 (c) $\widehat{\phi}_a(b[[\xi]]) \ll \widehat{\phi}_a(b[[\rho]])$.
Subcase $a = 0$, $b \,\eta\, Cr(a)$: $s[[1 \cdot n]] = b \cdot (Sn) \ll b \cdot (SSn) = s[[1 \cdot Sn]]$.
Subcase $a = a'\tilde{+}1$, $b = 0$: if $n = 0$ we have $\rho_0 = 0 \ll \widehat{\phi}_{a'} 0 = \rho_1$. In the step from n to Sn we have $\rho_n \ll \rho_{Sn} \prec \widehat{\phi}_{a'\tilde{+}1} 0$, therefore $\forall c \,\eta\, T'.\rho_n \prec c \prec \rho_{Sn} \to c \not\eta\, Cr(a')$, therefore $\rho_{Sn} = \widehat{\phi}_{a'} \rho_n \ll \widehat{\phi}_{a'} \rho_{Sn} = \rho_{SSn}$.
Subcase $a = a'\tilde{+}1$, $b = b'\tilde{+}1$: $\rho_0 = \widehat{\phi}_a b'$, $\forall c \,\eta\, T'.s[[\tau(s)^-]] \prec c \preceq s \to c \not\eta\, Cr(a)$. By lemma 9.30 (d) $\rho_0 = \widehat{\phi}_a b' \ll \widehat{\phi}_{a'} \widehat{\phi}_a b' = \rho_1$, and in the side induction step by lemma 9.30 (c) $\rho_{Sn} = \widehat{\phi}_{a'} \rho_n \ll \widehat{\phi}_{a'} \rho_{Sn} = \rho_{SSn}$.
Subcase $a = a'\tilde{+}1$, $b \,\eta\, Cr(a)$. By lemma 9.30 (d) $\rho_0 = b \ll \rho_1$, by lemma 9.18 (e) and in the side induction step follows, since $\forall x \,\eta\, T'.\rho_0 \preceq \rho_n \prec x \preceq s \to x \not\eta\, Cr(a)$, by lemma 9.30 (c) $\rho_{Sn} = \widehat{\phi}_{a'} \rho_n \ll \widehat{\phi}_{a'} \rho_{Sn} = \rho_{SSn}$.
Subcase $a \,\eta\, Lim' \setminus G'$, $b = 0$: Then $\forall c \,\eta\, T'.a^* \prec c \preceq a \to c \not\eta\, G'$ by lemma 9.27 (b), by IH $a^* \preceq a[[\tau(a)^-]] \preceq a[[\xi]] \ll a[[\rho]] \prec a$, therefore by lemma 9.30 (d) $s[[\xi]] = \widehat{\phi}_{a[[\xi]]} 0 \ll s[[\rho]]$.
Subcase $a \,\eta\, G'$, $b = 0$: $a[[\xi]] \ll a[[\rho]]$, $a \,\eta\, Cr(a[[\rho]])$, therefore $\widehat{\phi}_{a[[\xi]]} a \ll \widehat{\phi}_{a[[\rho]]} a$.

Subcase $a\ \eta\ Lim'$, $b = b'\tilde{+}1$. $a[[\xi]] \ll a[[\rho]]$, $\widehat{\phi}_a b'\ \eta\ Cr(a[[\rho]])$, therefore $\widehat{\phi}_{a[[\xi]]}(\widehat{\phi}_a b') \ll \widehat{\phi}_{a[[\rho]]}(\widehat{\phi}_a b')$.
Subcase $a\ \eta\ Lim'$, $b\ \eta\ Cr(a)$: $a[[\xi]] \ll a[[\rho]]$, $\widehat{\phi}_{a[[\xi]]} b \ll \widehat{\phi}_{a[[\rho]]} b$.
Case $s = D_a b$.
Subcase $a \neq I$, $b = 0$: By lemma 9.30 (e) $\rho_0 = a^- \ll \widehat{\phi}_{a^-} 0 = \rho_1$, and in the Step from n to Sn follows from $\rho_n \ll \rho_{Sn}$, for $a^- \preceq \rho_n \prec c \preceq \rho_{Sn} \prec D_a 0$ by lemma 9.19 (a) $c \not\eta\ G'$, $\widehat{\phi}_{\rho_n} 0 \ll \widehat{\phi}_{\rho_{Sn}} 0$.
Subcase $a = I$, $b = 0$: $\forall c\ \eta\ T'.c \prec s \rightarrow c \not\eta\ Fi'$. $\rho_0 = 0 \ll \widehat{\Omega}_0 = \rho_1$ by lemma 9.30 (a) and in the induction step follows from $\rho_n \ll \rho_{Sn} \prec s$, since $\forall c\ \eta\ T'.0 \prec c \preceq D_I 0 \rightarrow c \not\eta\ Fi'$, $\rho_{Sn} = \widehat{\Omega}_{\rho_n} \ll \widehat{\Omega}_{\rho_{Sn}} = \rho_{SSn}$.
Subcase $a \neq I$, $b = b'\tilde{+}1$: By lemma 9.19 (a) $\forall c\ \eta\ T'.\rho_0 = D_a b' \prec c \prec s \rightarrow c \not\eta\ G'$. By lemma 9.30 (d) $\rho_0 \ll \widehat{\phi}_{\rho_0} 0 = \rho_1$, and in the Step from n to Sn follows $\rho_n \ll \rho_{Sn}$, $\rho_{Sn} = \widehat{\phi}_{\rho_n} 0 \ll \widehat{\phi}_{\rho_{Sn}} 0 = \rho_{SSn}$.
Subcase $a = I$, $b = b'\tilde{+}1$: $\forall c\ \eta\ T'.\rho_0 = D_I b' \prec c \prec s \rightarrow c \not\eta\ Fi'$. $\rho_0 = D_I b' \ll \widehat{\Omega}_{D_I d'} = \rho_1$, and in the induction step follows from $\rho_n \ll \rho_{Sn} \prec s$ $\rho_{Sn} = \widehat{\Omega}_{\rho_n} \ll \widehat{\Omega}_{\rho_{Sn}} = \rho_{SSn}$.
Subcase $b\ \eta\ Lim'$, $\tau(b) \prec a$: $b[[\xi]] \ll b[[\rho]]$, therefore $D_a b[[\xi]] \ll D_a b[[\rho]]$.
Subcase $b\ \eta\ Lim'$, $a \preceq \tau(b) =: \pi$: We show $\zeta_n \ll \zeta_{Sn}$ by induction on $n : N$. Then $b[[\zeta_n]] \ll b[[\zeta_{Sn}]]$ by IH and $D_a(b[[\zeta_n]]) \ll D_a(b[[\zeta_{Sn}]])$.
Subsubcase $\pi \neq I \vee a \prec D_I b \vee a = I$. By lemma 9.30 (h) $\zeta_0 = \pi^- \ll D_\pi(b[[\zeta_0]])$ and in the induction step follows from $\zeta_n \ll \zeta_{Sn}$ $b[[\zeta_n]] \ll b[[\zeta_{Sn}]]$, $\zeta_{Sn} = D_\pi b[[\zeta_n]] \ll D_\pi b[[\zeta_{Sn}]] = \zeta_{SSn}$.
Subsubcase $\pi = I$, $D_I b \preceq a \prec I$. Then by lemma 9.20 and since $D_I b \preceq a \prec I$ follows $Dicom(a) = D_I c$ for some c and $D_I c \preceq a \prec D_I(c\tilde{+}1)$, therefore (by an immediate induction) $D_I c \preceq \zeta_n \prec \zeta_{Sn} \prec D_I(c\tilde{+}1)$, $\forall c\ \eta\ T'.\zeta_n \prec c \preceq \zeta_{Sn} \rightarrow c \not\eta\ Fi'$. Therefore$\zeta_0 = Dicom(a) \ll \widehat{\Omega}_{Dicom(a)} = \zeta_1$ and in the induction step follows from $\zeta_n \ll \zeta_{Sn}$ $\zeta_{Sn} = \widehat{\Omega}_{\zeta_n} \ll \widehat{\Omega}_{\zeta_{Sn}} = \zeta_{SSn}$.
Case $s = \widehat{\Omega}_a \vee s = I$: If $s[[z]] = z$ the assertion is trivial, and if $s[[z]] = \widehat{\Omega}_{a[[z]]}$ follows for $a^* \prec c \preceq a$ $c \not\eta\ Fi'$ since $a \not\eta\ Fi'$ and, since $a^* \preceq a[[\xi]] \ll a[[\rho]] \prec a$, follows by lemma 9.30 (h) $\widehat{\Omega}_{a[[\xi]]} \ll \widehat{\Omega}_{a[[\rho]]}$.

Lemma 9.32 *If $s, t, \xi\ \eta\ T'$,$\xi\tilde{+}1 \prec \tau(s)$, $s[[\xi]] \prec t \preceq s[[\xi\tilde{+}1]]$, then $s[[\xi]] \preceq t[[\tau(t)^-]]$.*

Proof:
$s[[\xi]] \ll s[[\xi\tilde{+}1]]$, therefore exist $s_0, \ldots, s_{Sn}$ such that $s_0 = s[[\xi]]$, $s_{Sn} = s[[\xi\tilde{+}1]]$ and $s_i = s^*_{Si}$. Then $s_i \prec t \preceq s_{Si}$ for some $i <_N n$, $s[[\xi]] \preceq s_i = s^*_{Si} \preceq t^* \preceq t[[\tau(t)^-]]$.

The most complicated task in this chapter is, to prove, that for $a, z\ \eta\ OT$, $z\ \eta\ \tau(a) \setminus \tau(a)^-$ we have $a[[z]]\ \eta\ OT$, and that $sup\{a[[z]]!z\ \eta\ (\tau(a) \setminus \tau(a)^-) \cap OT\} = a$ for Limes ordinals. To prove the latter in the case of $(D_a b)[[1 \cdot n]] = D_a(b[[\zeta_n]])$, we will argue, that by knowing $D_a b[[\zeta_0]] \prec D_a f \prec D_a b$, and $D_a f\ \eta\ OT$ we know $f \prec G_a f$, and have therefore some information on the components of f, which we want to relate to the components of b to have $D_a f \prec D_a b[[\zeta_n]]$ for some n. The components of $D_a b[[\zeta_n]]$ are built by iterated applications of the collaps and building $b[[z]]$. Therefore we need some relation, that interchanges in some way with the collapsing functions, allows to relate z to $a[[z]]$ and gives some informations on the G_π sets. To prove $a[[z]]\ \eta\ OT$ we need again such a relation.
The relation that solves this problem is $a \lhd_\xi b$, which relates $G_\pi a$ to the $G_\pi c$ for $a \preceq c \preceq b$, and allows to prove lemma 9.38, that relates $a[[z]]$ to z. Lemma 9.36 allows to infer from $D_\pi b\ \eta\ OT$ to $D_\pi a\ \eta\ OT$ in by $\lhd$-controlled situations. The relation interchanges with

some ordinal functions (see lemma 9.35). Theorem 9.39 and lemma 9.40 are the desired lemmata we need.

Definition 9.33 *Let* $s, t \;\eta\; T'$.
$s \lhd_\xi t :\Leftrightarrow s \prec t \wedge \forall \pi, r \;\eta\; T'.s \preceq r \preceq t \rightarrow G_\pi s \preceq G_\pi r \cup G^0_\pi \xi$,
and this is equivalent to $\forall \pi r \;\eta\; T'.s \prec r \preceq t \rightarrow G_\pi s \preceq G_\pi r \cup G^0_\pi \xi$.

Lemma 9.34 *Let* $a, b, c \;\eta\; T', \pi \;\eta\; R'$.

(a) $\forall x \;\eta\; G_\pi b.G_\pi x \subset G_\pi b \wedge length(x) <_N length(b)$.

(b) $a \prec D_\pi 0 \rightarrow G_\pi a \cong \emptyset$.

(c) $G_\pi \pi = 0$

(d) $(\tau(b) = \pi \;\eta\; R' \wedge G_\pi c \prec a \wedge c \prec b) \rightarrow c \prec b[[D_\pi a]]$.

(e) *If* $a \prec \pi$, $a \;\eta\; OT$, *then* $a \prec D_\pi b \leftrightarrow G_\pi a \prec b$.

(f) $\pi \preceq \rho \neq I \Rightarrow G_\rho a \subset G_\pi a$.

(g) $\pi \prec D_I a$, $G_\pi b \prec a$, $b \;\eta\; OT \Rightarrow G_I b \prec a$.

(h) *If* $\rho \;\eta\; R'$, $G_\pi \rho^- \cong G_\pi \rho$.

Proof:
(a) Immediate by induction on $length(a)$.
(b) Induction on $length(a)$.
The cases $a = b \tilde{+} c$ and $a = \widehat{\phi}_b c$ follow by IH. If $a = D_b c$ with $(b \neq I \vee b \preceq \pi \neq I)$, follows $b \prec \pi$, $G_\pi a \cong \emptyset$. If $a = D_I c$, $\pi \prec I$ follows $a \prec \pi$, $G_\pi a \cong \emptyset$. If $a = D_b c$, $b \prec I = \pi$ follows $G_\pi a = \emptyset$.
(d) Induction on $length(b)$, Side induction on $length(c)$:
Case $b \;\eta\; R'$: Then $b[[D_\pi a]] = D_\pi a$. If $c = d \tilde{+} e, \widehat{\phi}_d e$ follows the assertion by side IH for d, e.
Subcase $c = D_\xi e$. Then $\xi \preceq \pi \vee \xi = I$. If $\pi = \xi$ follows $e \;\eta\; G_\pi c \prec a$, $c \prec D_\pi a$, if $\xi \prec \pi \neq I$ follows $c \prec D_\pi a$, if $\xi \prec \pi = I$ follows $\xi \prec b$, $G_\pi \xi \cong G_\pi c \prec a$, therefore $\xi \prec D_\pi a$, $c \prec D_\pi a$, and if $\pi \prec \xi = I$ follows $D_\xi e \prec \pi \rightarrow D_\xi e \prec D_\pi b$.
Subcase $c = \widehat{\Omega}_e$: Then if $\pi \neq I$ follows from $c \prec \pi$, $c \prec D_\pi a$, and if $\pi = I$ follows by IH $e \prec D_\pi a$, $c \prec D_\pi a$.
Subcase $c = I, 0$: trivial.
Case $b = f \tilde{+} g$: if $c \prec f$ the assertion is trivial, otherwise $c = f \hat{+} d$ with $d \prec g$ and by IH follows the assertion.
Case $b = \widehat{\phi}_f g$, $b[[z]] = \widehat{\phi}_f (g[[z]])$. If $c = d \tilde{+} e$ the assertion follows by IH for d. If $c = \widehat{\phi}_d e$ follows if $d \prec f$ $e \prec b$, by IH $e \prec b[[D_\pi a]]$, $c \prec b[[D_\pi a]]$, if $f = d$ $e \prec g$, by IH $e \prec g[[D_\pi a]]$ and the assertion, if $f \prec d$ $c \preceq g$, $c \neq g$ since $g \not\eta\; Cr(f)$, $c \prec g[[D_\pi a]]$ and the assertion. If $c \;\eta\; G'$ follows $c \prec f$ or $c \prec g$ and therefore $c \prec g[[D_\pi a]]$, $c \prec b[[D_\pi a]]$.
Case $b = \widehat{\phi}_f g$, $b[[z]] = \widehat{\phi}_{f[[z]]} g'$, $g' = g = 0$ or $g' = f \;\eta\; G'$ and $g = 0$ or $g' = \widehat{\phi}_f g''$ and $g = g'' \tilde{+} 1$ or $g' = g \;\eta\; Cr(f)$. If $c = d \tilde{+} e$ follows the assertion by IH for d. If $c = \widehat{\phi}_d e$ follows, if $d \prec f$, $d \prec f[[D_\pi a]]$, further $e \prec b$, $e \prec b[[D_\pi a]]$, therefore $c \prec b[[D_\pi a]]$, if $d = f$ follows $e \prec g$, either $e \preceq g'' \wedge g'' \tilde{+} 1 = g$ therefore $c \prec g'$, $c \prec \widehat{\phi}_{f[[D_\pi a]]} g'$, or $e \prec g \;\eta\; Cr(f)$ therefore $c \prec g'$, $c \prec \widehat{\phi}_{f[[D_\pi a]]} g'$, and if $f \prec d$ follows $c \preceq g \preceq g'$, $c \preceq b[[D_\pi a]]$. If $c \;\eta\; G'$ follows by $c \prec b$ $c \preceq max_T\{f, g\}$, $g \prec c = f$ is not possible, therefore $c \preceq g$ or $c \prec f$, by IH $c \preceq max_T\{f[[D_\pi a]], g\}$ $c \prec a[[D_\pi a]]$.

Case $b = D_\rho f$, $\pi = \tau(f) \prec \rho$.
If $c = d\tilde{+}e, \widehat{\phi}_d e$ follows the assertion by side IH for d, e.
If $c = D_\xi d$ with $\xi \prec \rho \neq I$, the assertion is trivial.
If $c = D_\xi d$ with $\xi \prec \rho = I$ follows $\xi \prec D_\rho f$. If $\pi \preceq \xi$ or $\xi \prec \pi = I$ follows $G_\pi \xi \subset G_\pi D_\xi d \prec a$, if $\xi \prec \pi \neq I$ follows $\xi \preceq \pi^-$, $G_\pi \xi \cong \emptyset \prec a$, therefore by side IH $\xi \prec D_\rho f[[D_\pi a]]$, and the assertion.
If $c = D_I d$, $\rho \prec I$, follows $c \prec \rho$, $c \prec D_\rho f[[D_\pi a]]$.
If $c = D_\xi d$ with $\xi = \rho$ follows $G_\pi d \subset G_\pi c \prec a$, $d \prec f$, $d \prec f[[D_\pi a]]$ by IH, $c \prec b[[D_\pi a]]$.
If $c = \widehat{\Omega}_d$ follows, if $\rho \neq I$, $c \prec \rho$, $c \prec D_\rho f[[D_\pi a]]$, and if $\rho = I$ by IH $d \prec D_\rho f[[D_\pi a]]$, $c \prec D_\rho f[[D_\pi a]]$.
If $c = 0, I$, the assertion is trivial.
Case $b = \widehat{\Omega}_e$, $b \not\eta R'$.
If $c = d\tilde{+}e, \widehat{\phi}_d e$ follows the assertion by side IH.
If $c = D_\xi d$, $\xi \neq I$ follows $\xi \preceq b$, since $b \not\eta R'$, $\xi \prec b$. If $\pi \preceq \xi$ or $\xi \prec \pi = I$ follows $G_\pi \xi \subset G_\pi b \prec a$, and if $\xi \prec \pi \neq I$ follows $G_\pi \xi = \emptyset \prec a$, by side IH $\xi \prec b[[D_\pi a]]$.
If $c = D_I d$ follows $c \preceq e$, $c \neq e$, therefore $c \prec e[[D_\pi b]]$.
If $c = \widehat{\Omega}_d$, $d \prec f$ follows $G_\pi d \prec a$, by IH $d \prec f[[D_\pi a]]$, $c \prec b[[D_\pi a]]$.
If $c = 0, I$ the assertion ist trivial.
(e) "$\leftarrow$" follows by (d), with $b := \pi$. Proof of "$\rightarrow$" by Induction on $length(a)$.
If $a = c\tilde{+}d, \widehat{\phi}_b c, \widehat{\Omega}_c$, the assertion follows by IH, and if $a = 0, I$ the assertion is trivial.
Case $a = D_\rho c$.
If $\rho \prec \pi \neq I$ follows $G_\pi a \cong \emptyset$.
If $\rho \prec \pi = I$ follows $\rho \prec D_\pi b$, $G_\pi a \cong G_\pi \rho \prec b$ by IH.
If $\pi \prec \rho = I$ follows $a \prec D_\pi 0$, $G_\pi a \cong \emptyset$.
If $\rho = \pi$ follows $c \prec b$, $G_\pi a \cong G_\pi \pi \cup G_\pi c \cup \{c\} \prec b$, since $G_\pi c \cong G_\rho c \prec c$, $G_\pi \pi \cong \emptyset$.
(f) Induction on $length(a)$.
If $a = b\tilde{+}c, \widehat{\phi}_b d, \widehat{\Omega}_b$, the assertion follows by IH, and if $a = 0, I$, the assertion is trivial.
Case $a = D_\xi b$. If $\rho \preceq \xi \neq I$ follows $G_\rho a \cong \{b\} \cup G_\rho a \cup G_\rho \xi \subset \{b\} \cup G_\pi a \cup G_\pi \xi \cong G_\pi a$. If $\rho \preceq D_I b \wedge \xi = I$ follows $G_\rho a \cong \{b\} \cup G_\rho a \cup G_\rho \xi \subset \{b\} \cup G_\pi a \cup G_\pi \xi \cong G_\pi a$. Otherwise follows $G_\rho a \cong \emptyset$.
(g) Induction on $length(b)$.
If $b = c\tilde{+}d, \widehat{\phi}_c d, \widehat{\Omega}_c$ the assertion follows by IH, and if $b = 0, I$ the assertion is trivial.
Case $b = D_\rho c$.
If $\rho \prec \pi$ follows $b \prec \rho \preceq \pi \prec D_I a$, $G_I b \prec a$.
If $\pi \preceq \rho \prec I$ follows $G_\pi \rho \subset G_\pi b \prec a$, by IH $G_I d \cong G_I \rho \prec c$.
Subcase $\rho = I$: If $b \prec D_I a$ follows $c \prec a$ and, if $D_I a \prec b$, by $G_\pi b \cong G_\pi c \cup G_\pi I \cup \{c\} \prec a$ again $c \prec a$, therefore in both cases, since by $b \; \eta \; OT$, $G_I c \prec c$, $G_I b = G_I c \cup G_I I \cup \{c\} \preceq c \prec a$.
If $I \prec \rho$ follows $G_I d \cong \emptyset$.
(h) If $\rho = I$ this is trivial, if $\rho = \widehat{\Omega}_{c\tilde{+}1}$ follows $G_\pi \rho \cong G_\pi(c\tilde{+}1) \cong G_\pi c \cong G_\pi \rho$, and if $\rho = \widehat{\Omega}_c$ with $\rho^- = c$ follows $G_\pi \rho \cong G_\pi c$.

Lemma 9.35 *Let $a, b, c, \xi, \rho \; \eta \; T'$, $b \lhd_\xi c$.*

(a) $a \; \eta \; \{0, 1, \omega, \widehat{\Omega}_0, I\}$, $a \prec b \rightarrow a \lhd_0 b$.

(b) If $a \lhd_\xi c$, $a \preceq b \prec d \prec c$, $\forall \pi \; \eta \; R'.G_\pi b \preceq G_\pi a \cup G_\pi \xi^0$, then $b \lhd_\xi d$.

(c) If $a \prec c$, then $a \lhd_a c$.

(d) If $a \lhd_\xi b$, $\forall \pi \; \eta \; R'.G_\pi \xi \preceq G_\pi \rho$ (e.g. $\xi = 0$), then $a \lhd_\rho b$.

(e) *If* $c \neq 0$, *then* $a \lhd_\xi a\widehat{+}c$.

(f) $NF_+(a,c) \rightarrow a\widehat{+}b \lhd_\xi a\widehat{+}c$.

(g) $\forall d \,\eta\, T'.b \prec d \preceq c \rightarrow d \not\eta\, Cr(a)) \rightarrow \widehat{\phi}_a b \lhd_\xi \widehat{\phi}_a c$.

(h) $b \,\eta\, Cr(a) \cup \{0\}$, *then* $b \lhd_0 \widehat{\phi}_a b$.

(i) *If* $b = 0 \vee b \,\eta\, Cr(a)$ *and* $a' \lhd_\xi a$, *then* $\widehat{\phi}_{a'} b \lhd_\xi \widehat{\phi}_a b$.

(j) *If* $a \,\eta\, G'$, $a \lhd_0 \widehat{\phi}_a 0$.

(k) *If* $a \,\eta\, G'$, $a' \lhd_\xi a$, *then* $\widehat{\phi}_{a'} a \lhd_\xi \widehat{\phi}_a 0$.

(l) $a' \lhd_\xi a \rightarrow \widehat{\phi}_{a'}\widehat{\phi}_a b \lhd_\xi \widehat{\phi}_a(b\widetilde{+}1)$.

(m) $\widehat{\phi}_0 1 \lhd_0 \widehat{\phi}_1 0$.

(n) $D_a b \lhd_\xi D_a c$.

(o) $\widehat{\Omega}_b \lhd_\xi \widehat{\Omega}_c$.

(p) *If* $s \,\eta\, R'$, *then* $s^- \lhd_0 s$.

(q) $a^- \lhd_0 D_a b$.

Proof:
(a) Trivial, since $G_\pi a \cong \emptyset$.
(b) $b \preceq r \preceq d$, then $G_\pi b \preceq G_\pi a \cup G^0_\pi \xi \preceq G_\pi r \cup G^0_\pi \xi$.
(c), (d): trivial.
(e) If $a \preceq d \preceq a\widehat{+}c$ follows $d = a\widehat{+}e$ for some e and we have the assertion.
(f) $a\widehat{+}b \preceq d \preceq a\widehat{+}c$, then $d = a\widehat{+}e$ with $b \preceq e \preceq c$, $G_\pi b \preceq G_\pi e \cup G^0_\pi \xi$, $G_\pi(a\widehat{+}b) \preceq G_\pi a \cup G_\pi e \cup G^0_\pi \xi \cong G_\pi(a\widehat{+}e) \cup G^0_\pi \xi$.
(g) $s := \widehat{\phi}_a b \preceq d \preceq \widehat{\phi}_a c := t$. Then $d \not\eta\, Cr(a)$ since otherwise $b \prec d \preceq c$. We show $G_\pi s \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\widetilde{+}f$ follows $s \preceq max_T\{e,f\} \preceq \widehat{\phi}_a c$, $G_\pi s \preceq G_\pi max_T\{e,f\} \cup G^0_\pi \xi$. If $d = \widehat{\phi}_e f$ with $e \prec a$ follows $s \preceq f \preceq t$ and again the assertion by IH. If $d = \widehat{\phi}_a f$ follows $b \preceq f \preceq c$ and by $b \lhd_\xi c$ the assertion. $d \,\eta\, G'$ is not possible, otherwise $a \prec \widehat{\phi}_a b \prec d$, $d \,\eta\, Cr(a)$.
(h) The case $b = 0$ is trivial. Let therefore $b \,\eta\, Cr(a)$, $b \prec d \preceq \widehat{\phi}_a b$. Then $d \not\eta\, Cr(a)$ since otherwise $b \prec d \preceq b$. We show $G_\pi b \preceq G^0_\pi d$ by induction on $length(d)$. If $d = e\widetilde{+}f$ follows $b \preceq e \preceq \widehat{\phi}_a b$, $G_\pi b \preceq G^0_\pi e$. If $d = \widehat{\phi}_e f$ with $e \prec a$ follows $b \preceq f \preceq \widehat{\phi}_a b$ and again the assertion by IH. If $d = \widehat{\phi}_a f$ follows $b \preceq f \preceq b$, $b = f$ and the assertion trivially. $d \,\eta\, G'$ is not possible, otherwise $a \prec b \prec d$, $d \,\eta\, Cr(a)$.
(i) Let $\widehat{\phi}_{a'} b \prec d \preceq \widehat{\phi}_a b$. Then $d \not\eta\, Cr(a)$ since otherwise $b \prec d \preceq b$. We show $G_\pi \widehat{\phi}_{a'} b \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\widetilde{+}f$ this follows by IH. Case $d = \widehat{\phi}_e f$. If $e \prec a'$ follows $\widehat{\phi}_a b' \preceq f \preceq \widehat{\phi}_a b$ and by IH the assertion. If $a' \preceq e \preceq a$ follows $b \prec \widehat{\phi}_{a'} b \prec \widehat{\phi}_e f \preceq \widehat{\phi}_a b$ and by(h) $G_\pi b \preceq G_\pi \widehat{\phi}_e f \cup G^0_\pi \xi$, and with $a' \lhd_\xi a$, $G_\pi a' \preceq G^0_\pi e \cup G_\pi \xi$, therefore $G_\pi \widehat{\phi}_{a'} b \preceq G_\pi \widehat{\phi}_e f \cup G^0_\pi \xi$. $d \,\eta\, G' \cup \{0\}$ is not possible.
(j) Let $a \prec d \preceq \widehat{\phi}_a 0$. Then $d \not\eta\, Cr(a)$. We show $G_\pi a \preceq G_\pi d$ by induction on $length(d)$. If $d = e\widetilde{+}f$ this follows by IH. Case $d = \widehat{\phi}_e f$. If $e \prec a$ follows $a \preceq f \preceq \widehat{\phi}_a 0$ and by IH the assertion. If $e = a$ follows $G_\pi a \preceq G^0_\pi e$. $d \,\eta\, G'$ is not possible.
(k) Let $s := \widehat{\phi}_{a'} a \prec d \preceq \widehat{\phi}_a 0 =: t$. Then $d \not\eta\, Cr(a)$. We show $G_\pi \widehat{\phi}_{a'} a \preceq G^0_\pi d$ by induction on $length(d)$. If $d = e\widetilde{+}f$ this follows by IH. Case $d = \widehat{\phi}_e f$. If $e \prec a'$ follows $s \preceq f \preceq t$ and

by IH the assertion. If $a' \preceq e \preceq a$ follows $a \prec \widehat{\phi}_{a'}a \prec \widehat{\phi}_e f \preceq t$ and by (j) $G_\pi a \preceq G^0_\pi \widehat{\phi}_e f$, and with $a' \lhd_\xi a$, $G_\pi a' \preceq G_\pi e \cup G^0_\pi \xi$, therefore $G_\pi \widehat{\phi}_{a'} b \preceq G_\pi \widehat{\phi}_e f \cup G^0_\pi \xi$. $d \;\eta\; G' \cup \{0\}$ is not possible.
(l) Let $s := \widehat{\phi}_{a'}\widehat{\phi}_a b \prec d \preceq \widehat{\phi}_a(b\tilde{+}1) =: t$. Then $\widehat{\phi}_a b \prec d \preceq t$, therefore $d \not\eta\; Cr(a)$. We show $G_\pi s \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\tilde{+}f$ this follows by IH. Case $d = \widehat{\phi}_e f$. If $e \prec a'$ follows $s \preceq f \preceq t$ and by IH the assertion. If $a' \preceq e \preceq a$ follows $\widehat{\phi}_a b \prec \widehat{\phi}_{a'}\widehat{\phi}_a b \prec \widehat{\phi}_e f \preceq t$ and by (g), since $b \lhd_0 b\tilde{+}1$ $G_\pi \widehat{\phi}_a b \preceq G^0_\pi \widehat{\phi}_e f$, and with $a' \lhd_\xi a$, $G_\pi a' \preceq G_\pi e \cup G^0_\pi \xi$, therefore $G_\pi \widehat{\phi}_{a'}\widehat{\phi}_a b \preceq G_\pi \widehat{\phi}_e f \cup G^0_\pi \xi$. $d \;\eta\; G' \cup \{0\}$ is not possible.
(n) Let $s := D_a b \prec d \preceq D_a c =: t$. We show $G_\pi s \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\tilde{+}f$ or $d = \widehat{\phi}_e f$ follows $s \preceq max_T\{e,f\} \preceq t$, $G_\pi b \preceq G_\pi max_T\{e,f\} \cup G^0_\pi \xi$. Case $d = D_e f$: If $a \neq I \vee e = I$, follows $e = a$, $b \preceq f \preceq c$, and in all cases the assertion by $b \lhd_\xi c$, $b \preceq f$. If $a = I \wedge e \neq I$ follows $e \prec I$ and $s \preceq e \preceq t$, $G_\pi s = \emptyset$ or $\pi \preceq D_I b \preceq e$ or $\pi = I$ and $G_\pi s \preceq G_\pi e \cup G^0_\pi \xi \subset G_\pi(D_e f) \cup G^0_\pi \xi$.
Case $d = \widehat{\Omega}_e$. This is only possible, if $a = I$, $s \preceq e \preceq t$. Then by IH $G_\pi s \preceq G_\pi e \cup G^0_\pi \xi \cong G_\pi d \cup G^0_\pi \xi$.
Cases $d = 0, I$: not possible.
(m) $G_\pi \widehat{\phi}_0 1 \cong \emptyset$.
(o) Let $s := \widehat{\Omega}_b \prec d \preceq \widehat{\Omega}_c =: t$. We show $G_\pi s \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\tilde{+}f$, $\widehat{\phi}_e f$ follows $s \preceq max_T\{e,f\} \preceq t$, $G_\pi b \preceq G_\pi max_T\{e,f\} \cup G^0_\pi \xi$ by IH or trivially (if $s = max_T\{e,f\}$).
Case $d = D_e f$: If $e = I$, follows $b \prec d \preceq c$, $G_\pi s \cong G_\pi b \preceq G_\pi d \cup G^0_\pi \xi$. Otherwise $s \prec e \preceq t$. If in this case $\pi \preceq s \prec e$ or $e \preceq \pi = I$ follows $G_\pi s \preceq G_\pi e \cup G^0_\pi \xi \subset G_\pi D_e f \cup G^0_\pi \xi$. If $s \prec \pi \neq I$ follows $G_\pi s \cong \emptyset$.
Case $d = \widehat{\Omega}_e$. Then the assertion follows immediately by $b \prec e \preceq c$ and $b \lhd_\xi c$.
(p) If $s = \widehat{\Omega}_{a\tilde{+}1}$ this follows by $a \lhd_0 a\tilde{+}1$, $s^- = \widehat{\Omega}_a$ and (o). If $s \;\eta\; \{\widehat{\Omega}_0, I, \widehat{\Omega}_I\}$ follows $G_\pi(s) \cong \emptyset$.
Case $s = \widehat{\Omega}_{D_I b}$, let $s' := D_I b \prec d \preceq \widehat{\Omega}_{s'}$. We show $G_\pi s' \preceq G_\pi d \cup G^0_\pi \xi$ by induction on $length(d)$. If $d = e\tilde{+}f$, $\widehat{\phi}_e f$ follows $s' \preceq max_T\{e,f\} \preceq s$, $G_\pi s' \preceq G^0_\pi max_T\{e,f\}$ by IH or trivially (if $s = max_T\{e,f\}$).
Subcase $d = D_e f$: Then $e \neq I$, since otherwise $D_I b \prec d \preceq D_I b$. Therefore $s' \prec e \preceq s$, $e = t$. If now $\pi \preceq s = e$ or $s \prec \pi = I$, follows $G_\pi s \preceq G_\pi e \cup G^0_\pi \xi \subset G_\pi D_e f \cup G^0_\pi \xi$. If $s \prec \pi \neq I$, follows $s \prec D_\pi 0$, $G_\pi s \cong \emptyset$.
Case $d = \widehat{\Omega}_e$. Then $D_I b \preceq e \preceq D_I b$.
(q) Follows by (p), (b).

Lemma 9.36 *If $b \lhd_z a$, $G_\pi a \prec a$, $G_\pi z \prec b$, then $G_\pi b \prec b$.*

Proof:
Since $G_\pi b$ is finite and $\forall x, y \;\eta\; T'.x \prec y \vee x = y \vee y \prec x$, follows $G_\pi b \prec b \vee \exists c \;\eta\; G_\pi b.b \preceq c$. Assume $c \;\eta\; G_\pi b$ with minimal $length(c)$ such that $b \preceq c$. Since $G_\pi c \subset G_\pi b$ and for all $d \;\eta\; G_\pi c$, $length(d) <_N length(c)$ follows $G_\pi c \prec b$. Further $G_\pi b \preceq G_\pi a \cup G^0_\pi z \prec a$ therefore $b \preceq c \prec a$, by $b \lhd_z a$ follows $c \;\eta\; G_\pi b \preceq G_\pi c \cup G^0_\pi z \prec b$, a contradiction.

Lemma 9.37 *(a) If $s \neq 0$, then $s^* \lhd_0 s$.*

(b) $r \ll s \Rightarrow r \lhd_0 s$.

Proof: (a) By Induction on $length(s)$.
If $s \;\eta\; R'$ $s^- = s^* \lhd_0 s$.
If $s = a\tilde{+}b$ follows $b^* \lhd_0 b$, therefore the assertion by lemma 9.35 (f). If $s = \widehat{\phi}_a b$ follows the assertion by IH, lemma 9.35 (g) and lemma 9.27 (a), or lemma 9.35 (h), or lemma 9.35 (j), or lemma 9.35 (i) and IH or lemma 9.35 (m). If $s = D_a b$ follows the assertion by 9.35 (n) or 9.35 (q) and 9.35 (a). If $s = \widehat{\Omega}_a$, $s \not\eta\; R'$ follows by lemma 9.35 (o) the assertion.

(b) By (a).

Lemma 9.38 *Let $a\ \eta\ T'$.*

(a) If $z\ \eta\ \widehat{\tau(a)} \cap T'$, then $a[[z]] \lhd_z a$.

(b) $a \neq 0 \to a[[\tau(a)^-]] \lhd_0 a$.

Proof of (a), (b) simultaneously by Induction on $length(a)$. If $\tau(a) = \omega$ we show (except in some cases where the proof is trivial) by side induction on $n : N\ a[[1 \cdot n]] \lhd_0 a$, and (b) follows by (a).
If $a = 0$ this is trivial.
If $a\ \eta\ R'$, follows (a) by 9.35 (c), further $a[[\tau(a)^-]] = a^- \lhd_0 a$ by 9.35 (p).
Case $a = b\tilde{+}c$: by IH and 9.35 (f).
Case $a = \widehat{\phi}_b c$:
Subcase $b = c = 0$: trivial.
Subcase $b = 0$, $c = c'\tilde{+}1$: By lemma 9.35 (g) $\widehat{\phi}_0 c' \lhd_0 \widehat{\phi}_0 c$, further $G_\pi \widehat{\phi}_0 c' \cdot Sn = G_\pi \widehat{\phi}_0 c'$, by 9.35 (b) follows $\widehat{\phi}_0 c' \cdot SSn \lhd_0 \widehat{\phi}_0 c$.
Subcase $c\ \eta\ Lim'$, $c \not\eta\ Cr(b)$. $c[[z]] \lhd_z c$, $c[[\tau(c)^-]] \lhd_0 c$ by IH, $c^* \preceq c[[\tau(c)^-]] \preceq c[[z]] \prec c$, therefore by lemma 9.27 (a) and 9.35 (g) the assertion.
Subcase $b = 0$, $c\ \eta\ Cr(b)$: Lemma 9.35 (h) and (b).
Subcase $b = b'\tilde{+}1$, $c = 0$: By lemma 9.35 (i) $\rho_1 \lhd_0 a$, and $G_\pi a[[1 \cdot Sn]] \cong G_\pi \rho_1$, therefore $a[[1 \cdot n]] \lhd_0 a$.
Subcase $b = b'\tilde{+}1$, $c = c'\tilde{+}1$: By lemma 9.35 (g) $\rho_0 \lhd_0 a$, and $G_\pi a[[1 \cdot Sn]] \cong G_\pi \rho_0$, since $G_\pi a \cong G_\pi a' \cup G_\pi 0$, therefore $a[[1 \cdot n]] \lhd_0 a$.
Subcase $b = b'\tilde{+}1$, $c\ \eta\ Cr(b)$: By lemma 9.35 (i) $\rho_1 \lhd_0 a$, and $G_\pi a[[1 \cdot Sn]] \cong G_\pi \rho_1$, therefore $a[[1 \cdot n]] \lhd_0 a$.
Subcase $b\ \eta\ Lim'$, $c = 0, c'\tilde{+}1 \vee c\ \eta\ Cr(b)$: $b[[z]] \lhd_z b$, $b[[\tau(b)^-]] \lhd_0 b$, lemma 9.35 (i), (k), (l).
Case $a = D_b c$:
Subcase $c = 0$: By lemma 9.35 (q), (a) follows $\rho_0 \lhd_0 a$, and $G_\pi a[[1 \cdot n]] \cong G_\pi \rho_0$.
Subcase $c = c'\tilde{+}1$: By lemma 9.35 (n) follows $a[[0]] \lhd_0 a$, and $G_\pi a[[1 \cdot n]] \cong G_\pi a[[0]]$.
Subcase $c\ \eta\ Lim'$, $\tau(c) \prec b$: Lemma 9.35 (n) and IH.
Subcase $c\ \eta\ Lim'$, $b \preceq \tau(c)$:
Then $a[[1 \cdot n]] = D_b(c[[\zeta_n]])$. By IH we have $c[[\zeta_n]] \lhd_{\zeta_n} c$. Assume $(D_b c)[[1 \cdot n]] \preceq d \preceq D_b c$, $\pi\ \eta\ R'$. We show $G_\pi((D_b c)[[1 \cdot n]]) \preceq G^0_\pi d$ by side induction on $length(d)$.
If $d = e\tilde{+}f$ follows $G_\pi((D_b c)[[1 \cdot n]]) \preceq G^0_\pi e \subset G^0_\pi d$, if $d = \widehat{\phi}_e f$ follows the assertion similarly with e replaced by $max_T\{e, f\}$, and the case $d = \widehat{\Omega}_e$ is only possible if $b = I$, and the assertion follows by side IH for e.
Subsubcase $d = D_e f$, $e \neq b$: Then $b = I$, $a[[1 \cdot n]] \prec e \preceq a$. If $a[[1 \cdot n]] \prec \pi \neq I$ follows $G_\pi a[[1 \cdot n]] \cong \emptyset$, and if $\pi \preceq a[[1 \cdot n]] \vee \pi = I$ follows by IH $G_\pi a[[1 \cdot n]] \preceq G_\pi e \cup G^0_\pi \xi \subset G_\pi d \cup G^0_\pi \xi$.
Subsubcase $d = D_b f$: If $b \prec \pi \neq I$ or $D_b(c[[\zeta_n]]) \prec \pi \prec I \wedge b = I$ follows $G_\pi a[[1 \cdot n]] \cong \emptyset$. If $b \prec \pi = I$ follows $G_\pi a[[1 \cdot n]] \cong G_\pi b \cong G_\pi d$. Therefore assume $\pi \preceq b \neq I \vee (b = I \wedge (\pi \preceq D_I c[\zeta_n] \vee \pi = I))$. Then $G_\pi a[[1 \cdot n]] \cong G_\pi b \cup G_\pi c[[\zeta_n]] \cup \{c[[\zeta_n]]\}$, and, since $c[[\zeta_n]] \preceq f$, $G_\pi d \cong G_\pi b \cup G_\pi f \cup \{f\}$.
We show by side induction on $m : N$

$$(+) \qquad \forall m <_N n. G_\pi c[[\zeta_m]] \preceq G^0_\pi d.$$

We have by main IH $c[[\zeta_m]] \lhd_{\zeta_m} c \wedge c[[\zeta_m]] \preceq c[[\zeta_n]] \preceq f \preceq c$, therefore

$$(*) \qquad G_\pi c[[\zeta_m]] \preceq G_\pi f \cup G^0_\pi \zeta_m \subset G_\pi d \cup G^0_\pi \zeta_m \subset G_\pi d \cup G^0_\pi \zeta_m$$

Case $m = 0$: If $\tau(c) = I \wedge D_I c \preceq b \prec I$ follows $G_\pi \zeta_0 = G_\pi Dicom(b) \subset G_\pi b \subset G_\pi d$ by lemma 9.20 (b) and by $(*)$ follows the assertion, otherwise, we have $\zeta_0 = \tau(c)^-$, $c[[\zeta_0]] \lhd_0 c \wedge c[[\zeta_0]] \preceq f \preceq c$, therefore $G_\pi c[[\zeta_m]] \preceq G_\pi^0 f \cup G_\pi b$.
Case $m = Sm'$: If $\tau(c) \neq I$ or $b \prec D_I c$ or $b = I$ follows with $t := \tau(c)$:

$$\begin{aligned} G_\pi c[[\zeta_m]] &\preceq G_\pi f \cup G_\pi^0 \zeta_m \\ &\cong G_\pi f \cup G_\pi^0 D_t c[[\zeta_{m'}]] \\ &\preceq G_\pi f \cup G_\pi^0 t \cup G_\pi^0 c[[\zeta_{m'}]] \cup \{c[[\zeta_{m'}]]\} \\ &\preceq G_\pi f \cup G_\pi^0 d \cup G_\pi^0 c[[\zeta_{m'}]] \cup \{c[[\zeta_{m'}]]\} \\ &\preceq G_\pi^0 f \cup G_\pi^0 d \cup \{f\} \\ &\cong G_\pi^0 d \end{aligned}$$

since $\pi \preceq b \preceq \tau(c)$ and $G_\pi t \cong G_\pi \zeta_0 \preceq G_\pi(c[[\zeta_0]]) \preceq G_\pi^0 d$ by lemma 9.23 (b) or $\tau(c) = I$ and $G_\pi t \cong \emptyset$.
If $D_I c \preceq b \prec \tau(c) = I$ follows $G_\pi^0 \zeta_m \cong G_\pi^0 \zeta_0$, $G_\pi c[[\zeta_m]] \preceq G_\pi f \cup G_\pi^0 \zeta_0 \preceq G_\pi^0 d$.
Now we can conclude $(+)$ $G_\pi a[[1 \cdot n]] \cong G_\pi b \cup G_\pi c[[\zeta_n]] \cup \{c[[\zeta_n]]\} \preceq G_\pi b \cup G_\pi^0 d \cup \{f\} \cong G_\pi^0 d$.
Case $a = \widehat{\Omega}_b$, $a \not\eta R'$. Then by IH $b[[z]] \lhd_z b$, $b[[\tau(b)^-]] \lhd_0 b$ and by lemma 9.35 (o) follows the assertion.

Theorem 9.39 $\forall s, t \,\eta\, OT.s[[\tau(s)^-]] \preceq t \prec s \to \exists \xi \,\eta\, OT \cap \widehat{\tau(s)}.s[[\xi]] \preceq t \prec s[[\xi \tilde{+} 1]]$.

Proof by Induction on $length(s)$, side induction on $length(t)$. If $\tau(s) = \omega$ it is sufficient to prove $\exists n \in N.t \prec s[[1 \cdot n]]$, or (if $s[[1 \cdot n]] = \rho_{S^i n}$ to prove $\exists n \in N.t \prec \rho_n$) and for the minimal such $n : N$ follows the assertion.
Case $s = 0$: trivial.
Case $s \,\eta\, R'$: Let $\xi = t$.
Case $s = b \hat{+} c$. Then $b \hat{+} (c[[\tau(c)^-]]) \preceq t \prec b \hat{+} c$, $t = b \hat{+} d$ with $c[[\tau(c)^-]] \preceq d \prec c$, by IH $c[[\xi]] \preceq d \prec c[[\xi \tilde{+} 1]]$ for some $\xi \,\eta\, OT \cap \widehat{\tau(c)}$ and with this ξ we have the assertion.
Case $s = \widehat{\phi}_b c$: Then by lemma 9.27 (a) and (b) $t \not\eta\, Cr(b) \cup G'$
Subcase $b = c = 0$: Then $t = 0$, let $\xi = 0$.
Subcase $b = 0$, $c = c' \tilde{+} 1$: Then $\widehat{\phi}_0 c' \prec t \prec \widehat{\phi}_0 c$. If $t = \widehat{\phi}_d e$ follows by $t \not\eta\, Cr(0)$, $d = 0$, $c' \prec t \prec c$, which is not possible, therefore $t = d \tilde{+} e$. Then by IH or trivially we have $d \prec s[[1 \cdot n]]$ for some $n : N$, $e \preceq t \prec \widehat{\phi}_0 c$, $e \preceq \widehat{\phi}_0 c'$, $t \prec s[[1 \cdot Sn]]$.
Subcase $b = 0$, $c \,\eta\, Cr(b)$: Similar to the case $c = c' \tilde{+} 1$.
Subcase $c \,\eta\, Lim' \setminus Cr(b)$: If $t = d \tilde{+} e$ follows the assertion by side IH for d. If $t = \widehat{\phi}_d e$ and $d \prec b$ follows the assertion by side IH for e, and if $d = b$ follows by main-IH $d[[\xi]] \preceq e \prec d[[\xi \tilde{+} 1]]$ and the assertion.
Subcase $b = b' \tilde{+} 1$, $c = 0$: If $t = d \tilde{+} e$ follows the assertion by side IH for d. Subsubcase $t = \widehat{\phi}_d e$: If $d \prec b'$ follows the assertion by side IH for e. If $d = b'$ follows by side IH (or trivially if $e \prec \rho_0$) $e \prec \rho_n$, $\widehat{\phi}_d e \prec \widehat{\phi}_{b'} \rho_n = \rho_{Sn}$. If $d = b$ follows $e \prec 0$, a contradiction.
Subcase $b = b' \tilde{+} 1$, $c = c' \tilde{+} 1$: If $t = d \tilde{+} e$ follows the assertion by side IH for e. Subsubcase $t = \widehat{\phi}_d e$: If $d \prec b'$ follows the assertion by side IH for e. If $d = b'$ follows by side IH (or trivially if $e \prec \rho_0$) $e \prec \rho_n$, $\widehat{\phi}_d e \prec \widehat{\phi}_{b'} \rho_n = \rho_{Sn}$. If $d = b$ follows $e = c'$, let $\xi = 0$.
Subcase $b = b' \tilde{+} 1$, $c \,\eta\, Cr(b)$: As the last subcase.
Subcase $b \,\eta\, Lim' \setminus G'$, $c = 0$: The assertion follows by IH if $t = d \tilde{+} e$ or $t = \widehat{\phi}_d e$ with $d \prec b[[\tau(b)^-]]$. Otherwise follows $t = \widehat{\phi}_d e$ with $b[[\tau(b)^-]] \preceq d \prec b$, $b[[\xi]] \preceq d \prec b[[\xi \tilde{+} 1]]$ for some $\xi \,\eta\, OT \cap \tau(b)$. By $t \prec b$ follows $e \prec \widehat{\phi}_b 0$. If $e \prec \widehat{\phi}_{b[[\xi \tilde{+} 1]]} 0$ follows $s[[\xi]] \preceq t \prec s[[\xi \tilde{+} 1]]$. Otherwise, $s[[\tau(s)^-]] \preceq s[[\xi \tilde{+} 1]] \preceq e \prec s$, by side IH $s[[\rho]] \preceq e \prec s[[\rho \tilde{+} 1]]$ for some $\xi \prec \rho \prec \tau(s)^-$. Since $d \prec b[[\rho \tilde{+} 1]]$ follows $s[[\rho]] \preceq t \prec s[[\rho \tilde{+} 1]]$.

Subcase $b \ \eta \ G'$, $c = 0$: The assertion follows by IH if $t = d\tilde{+}e$ or $t = \widehat{\phi}_d e$ with $d \prec b[[\tau(b)^-]]$. Otherwise follows $t = \widehat{\phi}_d e$ with $b[[\tau(b)^-]] \preceq d \prec b$, $b[[\xi]] \preceq d \prec b[[\xi\tilde{+}1]]$ for some $\xi \ \eta \ OT \cap \tau(b)$. $b \preceq s[[\tau(s)^-]] \preceq \widehat{\phi}_d e \prec \widehat{\phi}_b 0$, $d \prec b \ \eta \ G'$, therefore $b \preceq e \preceq \widehat{\phi}_b 0$. If $e \prec \widehat{\phi}_{b[[\xi\tilde{+}1]]} b$ follows $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$. Otherwise, $s[[\tau(s)^-]] \preceq s[[\xi\tilde{+}1]] \preceq e \prec s$, by side IH $s[[\rho]] \preceq e \prec s[[\rho\tilde{+}1]]$ for some $\xi \prec \rho$. Since $d \prec b[[\rho\tilde{+}1]]$ follows $s[[\rho]] \preceq t \prec s[[\rho\tilde{+}1]]$.
Subcase $b \ \eta \ Lim'$, $c = c'\tilde{+}1$: Again the only interesting case is $t = \widehat{\phi}_d e$, $b[[\tau(b)^-]] \preceq d \prec b$, $b[[\xi]] \preceq d \prec b[[\xi\tilde{+}1]]$ by IH. $\widehat{\phi}_{b[[\tau(b)^-]]}\widehat{\phi}_b c' \preceq \widehat{\phi}_d e \prec \widehat{\phi}_b c$, if $b[[\tau(b)^-]] = d$, $\widehat{\phi}_b c' \preceq e \prec c \preceq \widehat{\phi}_b c$, and if $b[[\tau(b)^-]] \prec d$, $\widehat{\phi}_b c' \preceq \widehat{\phi}_d e \prec \widehat{\phi}_b c$ and, because $d \prec b$, $\widehat{\phi}_b c' \preceq e \prec \widehat{\phi}_b c$, in any case therefore $\widehat{\phi}_b c' \preceq e \prec \widehat{\phi}_b c$. If $e \prec s[[\xi\tilde{+}1]]$ follows $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$. Otherwise, $s[[\tau(s)^-]] \preceq s[[\xi\tilde{+}1]] \preceq e \prec s$, by side IH $s[[\rho]] \preceq e \prec s[[\rho\tilde{+}1]]$ for some $\xi \prec \rho$. Since $d \prec b[[\rho]]$ follows $s[[\rho]] \preceq t \prec s[[\rho\tilde{+}1]]$.
Subcase $b \ \eta \ Lim'$, $c \ \eta \ Cr(b)$: Again the only interesting case is $t = \widehat{\phi}_d e$, $b[[\tau(b)^-]] \preceq d \prec b$, $b[[\xi]] \preceq d \prec b[[\xi\tilde{+}1]]$ by IH. $\widehat{\phi}_{b[[\tau(b)^-]]} c \preceq \widehat{\phi}_d e \prec \widehat{\phi}_b c$, therefore $c \preceq e \prec \widehat{\phi}_b c$. If $e \prec s[[\xi\tilde{+}1]]$ follows $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$. Otherwise, $s[[\tau(s)^-]] \preceq s[[\xi\tilde{+}1]] \preceq e \prec s$, by side IH $s[[\rho]] \preceq e \prec s[[\rho\tilde{+}1]]$ for some $\xi \prec \rho$. Since $d \prec b[[\rho]]$ follows $s[[\rho]] \preceq t \prec s[[\rho\tilde{+}1]]$.
Case $s = D_b c$: Then $t \ \eta \ R'$ only possible if $b = I$.
Subcase $c = 0$, $b \neq I$. If $t = d\tilde{+}e$ follows from the side IH for d or trivially $d \prec \rho_n$ for some n, $t \prec \rho_{max_T\{n,1\}}$. If $t = \widehat{\phi}_c d$ follows $c, d \prec s$, $\exists n \in N.c, d \prec \rho_n$, $\widehat{\phi}_c d \prec \widehat{\phi}_{\rho_n} 0$. $t = D_d e, \widehat{\Omega}_e, I$ is not possible.
Subcase $c = 0$, $b = I$: If $t = d\tilde{+}e, \widehat{\phi}_d e$ follows by side IH or trivially $d, e \prec \rho_n$ for some $n : N$, $t \prec \rho_n$. If $t = D_d e$ follows $a \neq I$, $d \prec \rho_n$ for some n, $t \prec \rho_n$. If $t = \widehat{\Omega}_e$ follows $e \prec s$, $e \prec \rho_n$ for some $n : N$, $t \prec \rho_{Sn}$.
Subcase $c = c'\tilde{+}1$, $b \neq I$: If $t = d\tilde{+}e$ follows $d \prec \rho_n$ for some n, $t \prec \rho_n$. If $t = \widehat{\phi}_c d$ follows $c, d \prec s$, $c, d \prec \rho_n$ for some $n : N$, $\widehat{\phi}_c d \prec \widehat{\phi}_{\rho_n} 0$. If $t = D_d e$ follows $t = \rho_0$, and $t = \widehat{\Omega}_e, I$ is not possible.
Subcase $c = c'\tilde{+}1$, $b = I$: If $t = d\tilde{+}e, \widehat{\phi}_d e$ follows by side IH or trivially $d, e \prec \rho_n$ for some $n : N$, $t \prec s[[1 \cdot n]]$. If $t = D_d e$ follows $d = I$ and $t = \rho_0$ or $d \prec I$, $d \prec \rho_n$, $d \prec \rho_n$ for some n, $t \prec \rho_n$. If $t = \widehat{\Omega}_e$ follows $e \prec s$, $e \prec \rho_n$ for some $n : N$, $t \prec \rho_{Sn}$.
Subcase $c \ \eta \ Lim'$, $\tau(c) \prec b$: If $t = d\tilde{+}e$ follows by side IH $s[[\xi]] \preceq d \prec s[[\xi\tilde{+}1]]$, $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$, and if $t = \widehat{\phi}_d e$ follows similarly the assertion by side IH for $max_T\{d, e\}$. If $t = D_d e$ follows, if $b \neq I \vee d = b$, first $d = b$, by IH $c[[\xi]] \preceq e \prec c[[\xi\tilde{+}1]]$, $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$, and if $b = I$, $d \neq I$, $s[[\xi]] \preceq d \prec s[[\xi\tilde{+}1]]$, $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$. If $t = \widehat{\Omega}_d$ follows $b = I$, $s[[\xi]] \preceq d \prec s[[\xi\tilde{+}1]]$, and further $s[[\xi]] \preceq t \prec s[[\xi\tilde{+}1]]$.
Subcase $c \ \eta \ Lim'$, $b \preceq \tau(c) =: \pi$. If $t = 0, I$ the assertion follows in all cases trivially, if $t = d\tilde{+}e \vee t = \widehat{\phi}_d e \vee (t = D_d e \wedge d \neq b) \vee (t = \widehat{\Omega}_d \wedge b = I)$ by side IH. Let $t = D_b e$.
Subsubcase $\pi \neq I \vee b \prec D_I c \vee b = I$:
We show by induction on $length(f)$:

$$(*) \qquad G_\pi f \prec c \rightarrow \exists n \in N.G_\pi f \prec c[[\zeta_n]]$$

If $f = g\tilde{+}h, \widehat{\phi}_g h, \widehat{\Omega}_g$ follows $(*)$ by IH. Let $f = D_\xi h$.
If $\pi \preceq \xi \neq I \vee (\xi = I \wedge (\pi = I \vee \pi \preceq D_I b)$ follows $G_\pi \xi \cup G_\pi h \cup \{h\} \prec c$, by IH $G_\pi \xi \cup G_\pi h \prec c[[\zeta_n]]$ for some $n : N$. Further $G_\pi h \prec c[[\zeta_n]] \wedge \tau(c) = \pi \wedge h \prec c$, therefore by lemma 9.34 (d) $h \prec c[[D_\pi c[[\zeta_n]]]] = c[[\zeta_{Sn}]]$. Otherwise $G_\pi f = \emptyset$, or $G_\pi f = G_\pi \xi \prec c[[\zeta_n]]$ for some n.
Now by $t \ \eta \ OT$ follows $G_b e \prec e \prec c$ and since $b \preceq \pi \neq I \vee \pi = b \vee (\pi = I \wedge b \prec D_I c)$ follows by lemma 9.34 (f) and (g) $G_\pi d \prec c$, by $(*)$ $\exists n \in N.G_\pi d \prec c[[\zeta_n]]$. $\tau(c) = \pi \wedge d \prec c$, therefore by lemma 9.34 (d) $d \prec c[[D_\pi c[[\zeta_n]]]] = c[[\zeta_{Sn}]]$, $t \prec D_b c[[\zeta_{Sn}]] = a[[1 \cdot Sn]]$.

Subsubcase $\pi = I \wedge (D_I c \preceq b \prec I)$: We show by Induction on $length(f)$

$$(*) \qquad G_b f \prec c \wedge f \prec I \rightarrow \exists n \in N. f \prec \zeta_n$$

If $f = g\tilde{+}h, \widehat{\phi}_g h, \widehat{\Omega}_g$ the assertion follows by IH, if $f = 0, I$ trivially.
Case $f = D_\rho g$, $\rho \neq I$: If $\rho \preceq b$ follows, since by lemma 9.20 (a) $b \prec \zeta_n$ for some n, $\rho \prec \zeta_n$. If $b \prec \rho \prec I$ follows $G_b \rho \subset G_b f \prec c \wedge \rho \prec I$, therefore $f \prec \rho \prec \zeta_n$ for some n.
Case $f = D_I g$. If $b \preceq f$, follows $G_b f \cong G_b I \cup G_b g \cup \{g\} \prec c$, $f \prec D_I c \preceq b$ a contradiction. Therefore $f \prec b \prec \zeta_n$ for some n by lemma 9.20 (a).
Now we have $c[[0]] = c[[\widehat{\Omega}_0]] \preceq c[[\zeta_0]] \preceq e \prec c$, by IH we have $c[[\xi]] \preceq e \prec c[[\xi\tilde{+}1]]$. By lemma 9.31 and 9.37 (b) follows $c[[\xi]] \lhd_0 c[[\xi\tilde{+}1]]$, therefore by lemma 9.23 (b) $G_b\xi \preceq G_b c[[\xi]] \preceq G_b^0 e \preceq e \prec c$, $\xi \prec I$, therefore by $(*)$ $\xi \prec \zeta_n$ for some n, $t = D_I d \prec D_I d[[\xi\tilde{+}1]] \prec D_I d[[\zeta_{Sn}]] = s[[1 \cdot Sn]]$.

Lemma 9.40 *If $a \;\eta\; OT$, $z \;\eta\; OT \cap \widehat{\tau(a)}$, then $a[[z]] \;\eta\; OT$.*

Proof: Induction on $length(a)$.
Case $a = 0$: Obvious.
Case $a = b\tilde{+}c$. By IH $c[[z]] \;\eta\; OT$, $c[[z]] \prec c$, $b\hat{+}c \;\eta\; OT$, therefore by lemma 9.13 (h) $b\hat{+}c[[z]] \;\eta\; OT$.
Case $a = \widehat{\phi}_b c$: The assertion follows in all cases by IH, in some cases by induction on $length(z)$ (if $\tau(a) = \omega$).
Case $a = D_\pi c$:
Subcase $c = 0$: Obvious, since $\pi^- \;\eta\; OT$.
Subcase $c = c'\tilde{+}1$: Since $D_\pi(c'\tilde{+}1) \;\eta\; T'$ follows $G_\pi(c'\tilde{+}1) \prec (c'\tilde{+}1)$, $c' \lhd_0 c'\tilde{+}1$, by lemma 9.36 therefore $G_\pi c' \prec c'$, therefore $a[[0]] \;\eta\; OT$ (since $c', \pi \;\eta\; OT$). We conclude $a[[1 \cdot n]] \;\eta\; OT$ by induction on $n : N$.
Subcase $\omega \preceq \tau(c) \prec \pi$: $c[[z]] \;\eta\; OT$, $c[[z]] \lhd_z c$, $G_\pi c \prec c$. $z \prec \tau(c) \prec \pi$, by lemma 9.15 (h) $z \prec D_\pi z \preceq D_\pi c[[z]]$, by lemma 9.34 (e), since $z \;\eta\; OT$ $G_\pi z \prec c[[z]]$, therefore by lemma 9.36 $G_\pi c[[z]] \prec c[[z]]$, $a[[z]] \;\eta\; OT$.
Subcase $\pi \preceq \tau(c) =: t$, $t \neq I \vee \pi \prec D_I c \vee \pi = I$: We show by induction on m

$$(*) \qquad \forall m \leq_N n. G_\pi c[[\zeta_m]] \prec c[[\zeta_m]] \wedge G_t c[[\zeta_m]] \prec c[[\zeta_m]] \wedge c[[\zeta_m]] \;\eta\; OT$$

If $m = 0$ follows $c[[\zeta_0]] = c[[\tau(c)^-]] \lhd_0 c$, $G_\pi c \prec c$, by lemma 9.34 (f) and (g) $G_t c \prec c$ and therefore therefore by lemma 9.36 $G_\pi c[[\zeta_0]] \prec c[[\zeta_0]]$ and $G_t c[[\zeta_0]] \prec c[[\zeta_0]]$.
If $m = Sm'$, follows by IH $c[[\zeta'_m]] \;\eta\; OT$, $G_\pi c[[\zeta_{m'}]] \prec c[[\zeta_{m'}]]$ and $G_t c[[\zeta_{m'}]] \prec c[[\zeta_{m'}]]$ and therefore $\zeta_{Sm'} \;\eta\; OT$, $c[[\zeta_m]] \;\eta\; OT$. Further $c[[\zeta_{Sm'}]] \lhd_{\zeta_{Sm'}} c$, $G_\pi \zeta_{Sm'} \cong G_\pi D_t c[[\zeta_{m'}]] \subset G_\pi t \cup G_\pi c[[\zeta_{m'}]] \cup \{c[[\zeta_{m'}]]\}$. $G_\pi t \cong G_\pi \tau(c) \cong G_\pi \tau(c)^- \preceq G_\pi c[[\tau(c)^-]] \cong G_\pi c[[\zeta_0]] \prec c[[\zeta_0]] \preceq c[[\zeta_{m'}]]$, therefore $G_\pi \zeta_{Sm'} \preceq c[[\zeta_{m'}]] \prec c[[\zeta_{Sm'}]]$, $G_\pi c \prec c$, therefore $G_\pi c[[\zeta_{Sm'}]] \prec c[[\zeta_{Sm'}]]$. $G_t \zeta_{Sm'} \cong G_t D_t c[[\zeta_{m'}]] \subset G_t t \cup G_t c[[\zeta_{m'}]] \cup \{c[[\zeta_{m'}]]\} \preceq c[[\zeta_{m'}]] \prec c[[\zeta_m]]$, and we have $(*)$.
From $(*)$ follows $D_\pi c[[\zeta_n]] \;\eta\; OT$.
Subcase $D_I c \preceq \pi \prec I = \tau(c)$: $G_\pi \zeta_n \cong G_\pi \zeta_0 \cong G_\pi Dicom(c)$, $Dicom(c) = D_c d$ for some d, $D_I d \preceq c \prec D_I(d\tilde{+}1)$, $D_I d \lhd_0 D_I(d\tilde{+}1)$, therefore $Dicom(c) \lhd_0 c$, by $G_\pi c \prec c$ and lemma 9.36, $G_\pi Dicom(c) \prec Dicom(c)$. Since $c[[\zeta_n]] \lhd_{\zeta_n} c$, $G_\pi c \prec c$, $G_\pi \zeta_n = G_\pi \zeta_0 \prec \zeta_0 \preceq c[[\zeta_n]]$, follows $G_\pi c[[\zeta_n]] \preceq c[[\zeta_n]]$ by lemma 9.36. From $c \;\eta\; OT$ follows $Dicom(c) \;\eta\; OT$, therefore $\zeta_n \;\eta\; OT$ and it follows $D_\pi c[[\zeta_n]] \;\eta\; OT$.

Now we prove the property (F5), first the corresponding property for $\cdot[[\cdot]]$:

Lemma 9.41 *If $a, b \;\eta\; Lim$, $b \;\eta\; \widehat{\tau(a)}$ follows*

$$\tau(a[[b]]) = \tau(b) \wedge \forall \xi \;\eta\; \widehat{\tau(b)}. a[[b]][[\xi]] = a[[b[[\xi]]]]$$

(Note that $Lim \cong Lim' \cap OT$.

Proof:
Induction on $length(a)$. Since $b\ \eta\ Lim$, $\omega \prec \tau(a)$. Note that, if $a\ \eta\ Lim \leftrightarrow \tau(a)\ \eta\ Lim$, therefore in the situation of this lemma follows, if $\tau(z[\![b]\!]) = \tau(b)$, $z[\![b]\!]\ \eta\ Lim$.
If $a\ \eta\ R'$ the assertion is trivial.
If $a = c\tilde{+}d$ follows by IH
$\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = c\hat{+}(d[\![b[\![\xi]\!]]\!]) = c\hat{+}(d[\![b]\!][\![\xi]\!]) = (c\hat{+}d[\![b]\!])[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\phi}_c d$, $d\ \eta\ Lim \setminus Cr(c)$ follows, since $d^* \preceq d[\![\tau(d)^-]\!] \prec d[\![b]\!] \prec d$ by lemma 9.27 (a) $d[\![b]\!]\ \not\eta\ Cr(c)$, by IH $d[\![b]\!]\ \eta\ Lim$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\phi}_c(d[\![b]\!])) = \tau(d[\![b]\!]) = \tau(b)$, and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = \widehat{\phi}_c(d[\![b[\![y]\!]]\!]) = \widehat{\phi}_c(d[\![b]\!][\![\xi]\!]) = (\widehat{\phi}_c d[\![b]\!])[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\phi}_c 0$, $c\ \eta\ Lim \setminus G'$ follows, since $c^* \preceq c[\![\tau(c)^-]\!] \prec c[\![b]\!] \prec c$ by lemma 9.27 (b) $c[\![b]\!]\ \not\eta\ G'$, by IH $c[\![b]\!]\ \eta\ Lim$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\phi}_{c[\![b]\!]}0) = \tau(c[\![b]\!]) = \tau(b)$, and
$\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = \widehat{\phi}_{c[\![b[\![y]\!]]\!]}0 = \widehat{\phi}_{c[\![b]\!][\![\xi]\!]}0 = (\widehat{\phi}_{c[\![b]\!]}0)[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\phi}_c 0$, $c\ \eta\ G'$ follows $c\ \eta\ Cr(c[\![b]\!])$, by IH $c[\![b]\!]\ \eta\ Lim$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\phi}_{c[\![b]\!]}c) = \tau(c[\![b]\!]) = \tau(b)$, and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = \widehat{\phi}_{c[\![b[\![y]\!]]\!]}c = \widehat{\phi}_{c[\![b]\!][\![\xi]\!]}c =$
$\widehat{\phi}_{c[\![b]\!]}c)[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\phi}_c(d'\tilde{+}1)$, $c\ \eta\ Lim$ follows $\widehat{\phi}_c d'\ \eta\ Cr(c[\![b]\!])$, by IH $c[\![b]\!]\ \eta\ Lim$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\phi}_{c[\![b]\!]}\widehat{\phi}_c d') = \tau(c[\![b]\!]) = \tau(b)$, and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = \widehat{\phi}_{c[\![b[\![y]\!]]\!]}\widehat{\phi}_c d' =$
$\widehat{\phi}_{c[\![b]\!][\![\xi]\!]}\widehat{\phi}_c d' = (\widehat{\phi}_{c[\![b]\!]}\widehat{\phi}_c d')[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\phi}_c d$, $c\ \eta\ Lim$, $d\ \eta\ Cr(c)$ follows $d\ \eta\ Cr(c[\![b]\!])$, by IH $c[\![b]\!]\ \eta\ Lim$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\phi}_{c[\![b]\!]}d) = \tau(c[\![b]\!]) = \tau(b)$, and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = \widehat{\phi}_{c[\![b[\![y]\!]]\!]}d = \widehat{\phi}_{c[\![b]\!][\![\xi]\!]}d =$
$(\widehat{\phi}_{c[\![b]\!]}d)[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = D_\pi c$, $\tau(c) \prec \pi$, follows by IH $\tau(c[\![b]\!]) = \tau(b) \preceq b \prec \tau(c) \prec \pi$, $\tau(a[\![b]\!]) = \tau(D_\pi(c[\![b]\!])) = \tau(c[\![b]\!]) = \tau(b)$, and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] = D_\pi(c[\![b[\![y]\!]]\!]) = D_\pi(c[\![b]\!][\![\xi]\!]) =$
$(D_\pi(c[\![b]\!]))[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.
If $a = \widehat{\Omega}_c$, $a\ \not\eta\ R'$, follows by IH $c[\![b]\!]\ \eta\ Lim$, since $c^* \preceq c[\![\tau(c)^-]\!] \prec c[\![b]\!] \prec c$ by 9.27 (d) $c[\![b]\!]\ \not\eta\ Fi'$, therefore by IH $\tau(a[\![b]\!]) = \tau(\widehat{\Omega}_{c[\![b]\!]}) = \tau(c[\![b]\!]) = \tau(b)$ and $\forall\xi\ \eta\ \widehat{\tau(b)}.a[\![b[\![\xi]\!]]\!] =$
$\widehat{\Omega}_{c[\![b[\![y]\!]]\!]} = \widehat{\Omega}_{c[\![b]\!][\![\xi]\!]} = \widehat{\Omega}_{c[\![b]\!]}[\![\xi]\!] = a[\![b]\!][\![\xi]\!]$.

No we change from $\cdot[\![\cdot]\!]$ to $\cdot[\cdot]$. For the last property, we need to know something about $(\pi + a)[\![\xi]\!]$:

Lemma 9.42 *If $z\ \eta\ Lim' \cap T'$, $z \prec \pi\ \eta\ R' \cup \{\omega, 1\}$ follows*

$$\begin{array}{rcl} \tau(\pi^- + z) & = & \tau(z) \\ \wedge(& (& \forall\xi\ \eta\ \widehat{\tau(z)}.(\pi^- + z)[\![\xi]\!] = \pi^- + (z[\![\xi]\!])) \vee \\ & (& \tau(z) = \omega \wedge \forall\xi \prec \tau(z).(\pi^- + z)[\![\xi\tilde{+}1]\!] = \pi^- + (z[\![\xi]\!])). \end{array}$$

Proof:
Induction on $length(z)$.
If $z = a\tilde{+}b$ follows $\pi^- + z = (\pi^- + a)\tilde{+}b$, $\tau(\pi^- + z) = \tau(b) = \tau(z)$, $(\pi^- + z)[\![\xi]\!] = (\pi^- + a)\hat{+}b[\![\xi]\!] = \pi^- + (a\hat{+}b[\![\xi]\!]) = \pi^- + z[\![\xi]\!]$.
If $z\ \eta\ A$, $z \preceq \pi^-$ follows $(\pi^- + z)[\![\xi]\!] = (\pi^-\hat{+}z)[\![\xi]\!] = \pi^-\hat{+}(z[\![\xi]\!]) = \pi^- + (z[\![\xi]\!])$ by lemma 9.22 (c) and 9.13 (h).
If $\pi^- = 0$, the assertion is trivial, let therefore $\pi^- \neq 0$.

Case $\pi^- \prec z \,\eta\, A$: Then $\pi^- + z = z$. If in this case $z[\![\xi]\!] \,\eta\, A$ and $\pi^- \prec z[\![\tau(c)^-]\!]$ follows $(\pi^- + z)[\![\xi]\!] = z[\![\xi]\!] = \pi^- + (z[\![\xi]\!])$ and we have the assertion.
Subcase $z = \widehat{\phi}_a b$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \,\eta\, G'$, $\pi^- \preceq max_T\{a, b\}$, and if $\pi = \widehat{\Omega}_0$ follows $a = 0 \to 1 \prec b$.
$a = b = 0$ is not possible.
Subsubcase $a = 0$, $b = b'\tilde{+}1$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq b$, $\pi^- \preceq b'$, $(\pi^- + z)[\![1 \cdot n]\!] = \widehat{\phi}_0 b' \cdot SSn = \pi^- + (\widehat{\phi}_0 b' \cdot SSn) = \pi^- + (z[\![1 \cdot n]\!])$. If $\pi = \widehat{\Omega}_0$ follows, if $1 \prec b'$ $\pi^- \prec \widehat{\phi}_0 b'$ and the assertion as before, and if $b' = 1$, $(\pi^- + z)[\![n\hat{+}1]\!] = \widehat{\phi}_0 b' \cdot SSSn = \pi^- + (\widehat{\phi}_0 b' \cdot SSn) = \pi^- + (z[\![1 \cdot n]\!])$.
Subsubcase $a = 0$, $b \,\eta\, Cr(a)$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq b$, and if $\pi = \widehat{\Omega}_0$, we get $\pi^- \preceq b$ directly. If now $\pi^- = b$ follows $(\pi^- + z)[\![1 \cdot n\hat{+}1]\!] = b \cdot SSn = \pi^- + (b \cdot Sn) = \pi^- + (z[\![1 \cdot n]\!])$, and if $\pi^- \prec b$ follows $(\pi^- + z)[\![1 \cdot n]\!] = b \cdot Sn = \pi^- + (b \cdot Sn) = \pi^- + (z[\![1 \cdot n]\!])$.
In the cases $a \neq 0 \vee b \,\eta\, (Lim \setminus Cr(a))$ we have $z[\![\xi]\!] \,\eta\, A$ and we show $\pi^- \prec z[\![\tau(z)^-]\!]$.
Subsubcase $b \,\eta\, Lim \setminus Cr(a)$: If $\pi \neq \widehat{\Omega}_0$ follows, if $\pi^- \preceq a$, $\pi^- \prec \widehat{\phi}_a(b[\![\tau(b)^-]\!])$, and if $a \prec \pi^-$, $\pi^- \preceq b$, since $b \not\eta\, Cr(a)$ $\pi^- \prec b$, therefore by lemma 9.27 (a), since $\pi^- \,\eta\, Cr(a)$, $\pi^- \preceq b^* \preceq b[\![\tau(b)^-]\!] \prec z[\![\tau(z)^-]\!]$. If $\pi = \widehat{\Omega}_0$ follows if $0 \prec a$, $\pi^- \prec z[\![\tau(z)^-]\!]$, and if $a = 0$, $b \,\eta\, Lim$, therefore by lemma 9.23 (e) $1 \prec b[\![\tau(b)^-]\!]$, $\pi^- \prec z[\![\tau(z)^-]\!]$.
Subsubcase $a = a'\tilde{+}1$, $b = 0$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\} = a$, $\pi^- \preceq a' \prec z[\![0]\!]$. If $\pi = \widehat{\Omega}_0$ follows $\pi^- = \omega \prec z[\![0]\!]$.
Subsubcase $a = a'\tilde{+}1$, $b = b'\tilde{+}1$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\}$, $\pi^- \preceq max_T\{a, b'\}$, $\pi^- \prec z[\![0]\!]$. If $\pi = \widehat{\Omega}_0$ follows $\pi^- = \omega \prec z[\![0]\!]$.
Subsubcase $a = a'\tilde{+}1$, $b \,\eta\, Cr(a)$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\}$, $\pi^- \preceq max_T\{a', b\}$, $\pi^- \prec z[\![0]\!]$. If $\pi = \widehat{\Omega}_0$ follows, since $1 \prec b$, $\pi^- = \omega \prec z[\![0]\!]$.
Subsubcase $a \,\eta\, Lim \setminus G'$, $b = 0$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\} = a$, $\pi^- \neq a$, since $a \not\eta\, G'$, $\pi^- \prec a$, $\pi^- \preceq a[\![\tau(a)^-]\!] \prec z[\![\tau(z)^-]\!]$. If $\pi = \widehat{\Omega}_0$ follows, by lemma 9.23 (e) $1 \prec a[\![\tau(a)^-]\!]$, $\pi^- = \omega \prec z[\![\tau(z)^-]\!]$.
Subsubcase $a \,\eta\, G'$, $b = 0$: If $\pi = \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\} = a \prec z[\![\tau(z)^-]\!]$. If $\pi = \widehat{\Omega}_0$ follows, by lemma 9.23 (e) $1 \prec a[\![\tau(z)^-]\!]$, $\pi^- = \omega \prec z[\![\tau(z)^-]\!]$.
Subsubcase $a \,\eta\, Lim$, $b = b'\tilde{+}1$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\}$, $\pi^- \preceq max_T\{a, b'\}$, $\pi^- \prec \widehat{\phi}_a b' \prec z[\![\tau(z)^-]\!]$. If $\pi = \widehat{\Omega}_0$ follows the assertion immediately.
Subsubcase $a =\!\eta\, Lim$, $b \,\eta\, Cr(a)$: If $\pi \neq \widehat{\Omega}_0$ follows $\pi^- \preceq max_T\{a, b\} = b \prec z[\![\tau(z)^-]\!]$, and if $\pi = \widehat{\Omega}_0$ follows the assertion by $1 \prec b$.
Subcase $z = D_\rho b$. Then, since $I^- = 0$, $\rho = \pi \neq I$, $\pi^- \prec z[\![\tau(z)^-]\!]$ and $z[\![\xi]\!] \,\eta\, A$.
Subcase $z = \widehat{\Omega}_a$: Since $\pi^- \prec z \prec \pi$ follows $\pi = I$, $\pi^- = 0$, $(\pi^- + z)[\![\xi]\!] = \pi^- + (z[\![\xi]\!])$
Subcases $z = 0, I$: not possible.

Definition 9.43 *Now we define the fundamental sequences, as we needed them in chapter 8:*
$\cdot[\cdot] := \lambda x, y.x[\![\tau(x)^- + y]\!]$.
We write $r[s]$ for $(\cdot[\cdot])rs$.

Lemma 9.44 $\forall x, y \,\eta\, Lim.y \prec \tau(x) \to \tau(x[y]) = \tau(y) \wedge$
$(\forall \xi \prec \tau(y).x[y][\xi] = x[y[\xi]]) \vee$
$\tau(x) = \omega \wedge \forall n \,\eta\, \omega.x[y][n\tilde{+}1] = x[y[n]]$.

Proof:
$\tau(x[y]) = \tau(x[\![\tau(x)^- + y]\!]) = \tau(\tau(x)^- + y) = \tau(y)$ and for all $z \prec \tau(y)$ $x[y][z] = x[\![\tau(x)^- + y]\!][\![\tau(y)^- + z]\!] = x[\![(\tau(x)^- + y)[\![\tau(y)^- + z]\!]]\!] = x[\![\tau(x)^- + (y[\![\tau(y)^- + z]\!])]\!] = x[y[z]]$, or $\tau(y) = \omega$ and $x[y][z\tilde{+}1] = x[\![(\tau(x)^- + y)[\![\tau(y)^- + z\tilde{+}1]\!]]\!] = x[\![\tau(x)^- + (y[\![\tau(y)^- + z]\!])]\!] = x[y[z]]$.

Lemma 9.45 *OT fulfills all the properties of General Assumption 8.10 and 8.28*

Proof: This is all proven in this chapter or easy, only most properties of $\cdot[\cdot]$ are not immediately proven, but similar results are proven for $\cdot[\![\cdot]\!]$ and can be easily transferred by using

$$\forall x, z, z' \;\eta\; OT . \tau(x)^- \;\eta\; OT \wedge \tau(x)^- + z \;\eta\; OT \wedge (z \prec z' \rightarrow \tau(a)^- + z \prec \tau(a)^- + z')$$

Chapter 10

Comparison of OT with the ordinals in [Buc92b]

In this chapter we prove that the ordinal denotation system OT is in accordance with the functions we used in 7.8, which correspond to the system of [Buc92b]. We follow the lines of [Buc86]and [BS88]. In this chapter we will use T, OT $\prec$ etc. for the sets, functions and relations, which are defined in HA and correspond to the definitions used in chapter 9.
We introduce the ordinal functions as used in [Buc92b](definition 10.1, 10.2), and cite some lemmata (10.3, 10.4). Next we introduce $\widehat{\phi}$ and $\widehat{\Omega}$ as ordinal functions (definition 10.5), the interpretation of the ordinal denotations (definition 10.6). To prove the equivalence, we show that $C(\alpha, \beta)$ can be defined in a more restricted way (definition 10.8 and lemma 10.9), that we can invert functions like $+$ and Ω. (lemma 10.10), and that the ordinals are ordered as the denotations (lemma 10.12). Now we prove, that we can define $C(\alpha, \beta)$ in a way such that we allow $\psi_\pi\alpha$ only for $\alpha \in C_\pi(\alpha)$. Now we can define G_π on ordinals (definition 10.19), and conclude, that the interpretation of ordinal denotation is correct (lemmata10.21, 10.22 and 10.7).

We first repeat the definitions of [Buc92b]:
Definition 10.1 *(variant of definition 4.1 of [Buc92b]) Let # be the direct sum on ordinals.* $\Omega_0 := 0$, $\Omega_\sigma := \aleph_\sigma$ *for* $\sigma > 0$.
We assume the existence of a weakly inaccessible cardinal, e. g. a regular fixed point of $\sigma \mapsto \Omega_\sigma$ *and define*

$$I := min\{\sigma | \sigma regular\ Cardinal \wedge \Omega_\sigma = \sigma\}$$

$I^+ := sup\{\zeta_n | n < \Omega\}$, *where* $\zeta_0 := \Omega_{I+1}$, $\zeta_{n+1} := \Omega_{\zeta_n}$,

$$On := \{\alpha | \alpha\ ordinal\ , \alpha < I^+\}$$

$$R := \{\sigma \in On | \omega < \sigma \wedge \sigma regular\} = \{I\} \cup \{\Omega_{\sigma+1} | \sigma < I^+\}$$

Let κ, π, τ *denote elements of* R, $\alpha, \beta, \gamma, \delta$ *elements of* On.
Let ϕ *be the usual Veblen-function.*

Definition 10.2 *(variant on definition 4.1 of [Buc92b]) By transfinite recursion on* α, *we define ordinals* $\psi_\kappa\alpha$ *and sets* $C(\alpha, \beta) \subset On$ $(\kappa \in R)$ *as follows:*

$$\psi_\kappa\alpha := min\{\beta | \kappa \in C(\alpha, \beta) \wedge C(\alpha, \beta) \cap \kappa \subset \beta\}$$

$$C(\alpha,\beta) := \left\{ \begin{array}{l} \text{the closure of } \beta \cup \{0, I\} \text{ under the functions} \\ +,\ \phi,\ \sigma \mapsto \Omega_\sigma,\ (\xi,\pi) \mapsto \psi_\pi \xi\ (\xi < \sigma,\ \pi \in R) \end{array} \right.$$

(Note that by I.H. $\psi_\pi\xi$ is already defined for all $\xi < \alpha$, $\pi \in R$.)
We define $\psi_\kappa : On \to On$, $\psi_\kappa(\alpha) := \psi_\kappa\alpha$, $C_\kappa(\alpha) := C(\alpha, \psi_\kappa\alpha)$.

Lemma 10.3 *(Lemma 4.4 of [Buc92b])*

(a) $\beta < \pi \Rightarrow$ *cardinality*$(C(\alpha,\beta)) < \pi$

(b) $C(\alpha,\beta) = \bigcup_{\eta<\beta} C(\alpha,\eta)$, *for each limit ordinal* β.

(c) $\kappa \in C(\alpha,\kappa)$.

(d) $C_\kappa(\alpha) \cap \kappa = \psi_\kappa\alpha$.

Proof:
All statements are immediate consequences of definition 10.2.

Lemma 10.4 *(Lemma 4.5 of [Buc92b])*

(a) $\psi_\kappa\alpha < \kappa \wedge \psi_\kappa\alpha \notin C_\kappa(\alpha)$

(b) $(\alpha_0 < \alpha \wedge \alpha_0 \in C_\kappa(\alpha)) \to \psi_\kappa\alpha_0 < \psi_\kappa\alpha$

(c) $(\alpha_0 < \alpha \wedge \alpha_0 \in C_\kappa(\alpha) \Rightarrow \psi_\kappa\alpha_0 < \psi_\kappa\alpha$

(d) $\Omega_\sigma \in C(\alpha,\beta) \Rightarrow \sigma \in C(\alpha,\beta)$

(e) $\Omega^{\xi_0}\#\cdots\#\Omega^{\xi_n} \in C(\alpha,\beta) \Rightarrow \{\xi_0,\ldots,\xi_n\} \subset C(\alpha,\beta)$

(f) $\kappa = \Omega_{\sigma+1} \Rightarrow \Omega_\sigma < \psi_\kappa\alpha < \Omega_{\sigma+1}$

(g) $\Omega_{\psi_I\alpha} = \psi_I\alpha$

(h) $(\Omega_\sigma \le \gamma \le \Omega_{\sigma+1} \wedge \gamma \in C(\alpha,\beta)) \Rightarrow \sigma \in C(\alpha,\beta)$.

(i) $\alpha_0 \le \alpha \Rightarrow (\psi_\kappa\alpha_0 \le \psi_\kappa\alpha \wedge C_\kappa(\alpha_0) \subset C_\kappa(\alpha))$

Proof: See [Buc92b]

Definition 10.5 $A := \{\alpha \in On | \alpha \text{ principal additive number}\} =$
$\{\alpha \in On | \forall\beta,\gamma < \alpha.\beta + \gamma < \alpha\}$,
$Lim := \{\alpha \in On | \alpha \textit{Limes ordinal}\}$,
$Suc := \{\alpha + 1 | \alpha \in On\}$,
$G := \{\alpha \in On | \alpha \textit{Gamma ordinal}\} = \{\alpha \in On | \alpha = \phi_\alpha 0\}$,
$Fi := \{\alpha \in On | \alpha \textit{fixed point of } \sigma \mapsto \Omega_\sigma\} = \{\alpha \in On | \alpha = \Omega_\alpha\}$.
$Cr(\alpha) := \{\gamma \in On | \gamma = \phi_\alpha\gamma\}$ *For* $\alpha = \omega^{\alpha_1} +, \ldots, +\omega^{\alpha_n}$, $\alpha_1 \ge \cdots \alpha_n$, *we define:*

$last(\alpha) := \omega^{\alpha_n}$, $first(\alpha) := \omega^{\alpha_1}$, $length(\alpha) := n$.
$last(0) := first(0) := 0$, $length(0) := 0$.

$$\widehat{\phi}_\alpha\beta := \begin{cases} \phi_\alpha(\beta+1) & \text{if } \exists n < \omega, \gamma.\beta = \gamma + n \wedge \\ & \quad (\gamma = 0 \wedge \alpha \in G) \vee \gamma \in Cr(\alpha) \\ \phi_\alpha\beta & \text{otherwise} \end{cases}$$

$$\widehat{\Omega}_\alpha := \begin{cases} \Omega_{\alpha+1} & \text{if } \exists n < \omega, \beta.\alpha = \beta + n \wedge \beta \in Fi \cup \{0\}, \\ \Omega_\alpha & \text{otherwise.} \end{cases}$$

$\Omega^-_{\sigma+1} := \Omega_\sigma,\ I^- := 0.$

$NF_+(\alpha,\beta) :\Leftrightarrow \alpha \neq 0 \wedge \beta \in A \wedge \beta \leq last(\alpha).$
(Note, that this definition differs from definition 9.11 (i).

Definition 10.6 *For $a \in OT$ we define an ordinal $o(a) \in On$:*
$o(0) := 0,$
$o(I) := I,$
$o(a\tilde{+}b) := o(a) + o(b),$
$o(\widehat{\phi}_a b) := \widehat{\phi}_{o(a)} o(b),$
$o(\widehat{\Omega}_a) := \Omega_{o(a)},$
$o(D_a b) := \psi_{o(a)} o(b).$

We will prove the following lemma:

Lemma 10.7 *(a)* $\quad C_{\Omega_1}(I^+) = \{o(x) \mid x \in OT\}.$
(b) $\quad \forall a \in OT$ *such that* $a \prec \widehat{\Omega}_0$*:*
$\quad\quad o(a) = ordertype(\{x \in OT \mid x \prec a\}, \prec).$
(c) $\quad \psi_{\Omega_1} I^+ = ordertype\{x \in OT \mid x \prec \widehat{\Omega}_0\}, \prec).$

Proof: At the end of this chapter.

Definition 10.8 *Let $\alpha, \beta \in On$.*

$$\begin{array}{rcll} C^0(\alpha,\beta) & := & \beta \cup \{0, I\} & \\ C^{n+1}(\alpha,\beta) & := & C^n(\alpha,\beta) & \cup\ \{\widehat{\phi}_\gamma\delta, \widehat{\Omega}_\gamma | \gamma,\delta \in C^n(\alpha,\beta)\} \\ & & & \cup\ \{\gamma+\delta | \gamma,\delta \in C^n(\alpha,\beta), NF_+(\gamma,\delta)\} \\ & & & \cup\ \{\psi_\pi\xi | \pi,\xi \in C^n(\alpha,\beta),\ \pi \in R,\ \delta < \alpha\} \end{array}$$

$C^n_\tau(\alpha) := C^n(\alpha, \psi_\tau\alpha).$

Lemma 10.9 $\bigcup_{n<\omega} C^n(\alpha,\beta) = C(\alpha,\beta).$

Proof: "$\supset$": If $\delta \in C(\alpha,\beta)$, then $\delta + 1 \in C(\alpha,\beta)$, therefore $C^0(\alpha,\beta) \subset C(\alpha,\beta)$, if $\gamma,\delta \in C(\alpha,\beta)$, then $\widehat{\phi}_\gamma\delta \in \{\phi_\gamma\delta, \phi_\gamma(\delta+1)\} \subset C(\alpha,\beta)$, and $\widehat{\Omega}_\gamma \in \{\Omega_\gamma, \Omega_{\gamma+1}\} \subset C(\alpha,\beta)$, therefore $C^n(\alpha,\beta) \subset C(\alpha,\beta)$.
"$\supset$": By induction on n follows easily, if $\omega^{\gamma_1} + \cdots + \omega^{\gamma_n} \in C^n(\alpha,\beta)$, $\gamma_1 \geq \cdots \geq \gamma_n$, then $\omega^{\gamma_i} \in C^n(\alpha,\beta)$, especially $\gamma + 1 \in C^n\alpha,\beta) \to \gamma \in C^n(\alpha,\beta)$.
Therefore we have, if $\gamma,\delta \in C^n(\alpha,\beta)$, then $\phi_\gamma\delta \in \{\widehat{\phi}_\gamma\delta, \gamma, \delta\} \subset C^{n+1}(\alpha,\beta)$ or $\delta = \delta' + 1$ and $\phi_\gamma\delta = \widehat{\phi}_\gamma\delta' \in C^{n+1}(\alpha,\beta)$, in a similar way follows $\gamma \in C^n(\alpha,\beta) \to \Omega_\gamma \in C^{n+1}(\alpha,\beta)$.
If $\gamma,\delta \in C^n(\alpha,\beta)$,

$$\gamma = \omega^{\gamma_1} + \cdots + \omega^{\gamma_n},\ \gamma_1 \geq \cdots \geq \gamma_n,$$
$$\delta = \omega^{\delta_1} + \cdots + \omega^{\delta_m},\ \delta_1 \geq \cdots \geq \delta_m,$$

next

$$\gamma + \delta = \omega^{\gamma_1} + \cdots + \omega^{\gamma_i} + \omega^{\delta_1} + \cdots + \omega^{\delta_m} \in C^{n+i}(\alpha,\beta)$$

for some i.
Therefore we have "$\subset$".

Lemma 10.10 *(a) If $\gamma + \delta \in C(\alpha,\beta)$, $NF_+(\gamma,\delta)$, then $\gamma,\delta \in C(\alpha,\beta)$.*

(b) $\widehat{\phi}_\gamma\delta \in C(\alpha,\beta) \to \gamma,\delta \in C(\alpha,\beta)$.

(c) $\Omega_\gamma \in C(\alpha,\beta) \to \gamma \in C(\alpha,\beta)$.

(d) $\psi_\tau\gamma \in C(\alpha,\beta) \to \tau \in C(\alpha,\beta)$.

Proof: (a) - (c):
We have:
If $NF_+(\gamma,\delta)$, $NF_+(\gamma',\delta')$, $\gamma+\delta = \gamma'+\delta'$, then $\gamma=\gamma'$, $\delta=\delta'$. Further follows in this case $\gamma+\delta \notin A$, $0, I, \widehat{\phi}_\alpha\beta, \widehat{\Omega}_\alpha, \psi_\pi\rho \in A$.
If $\widehat{\phi}_\gamma\delta = \widehat{\phi}_{\gamma'}\delta'$, then $\gamma=\gamma'$, $\delta=\delta'$, further $\widehat{\phi}_\gamma\delta \notin G\cup\{0\}$, $I, \widehat{\Omega}_\alpha, \psi_\pi\rho \in G$.
If $\widehat{\Omega}_\gamma = \widehat{\Omega}_{\gamma'}$, then $\gamma=\gamma'$, further $\widehat{\Omega}_\gamma \in R\setminus\{I\}$, $\psi_\pi\rho \notin R$.
Now we prove by induction on n, that if $\gamma+\delta \in C^n(\alpha,\beta)$ γ,δ as before, then $\gamma,\delta \in C^n(\alpha,\beta)$: If $\gamma+\delta \in C^0(\alpha,\beta)$, then $\gamma,\delta < \gamma+\delta < \beta$, $\gamma,\delta<\beta$, and in the induction step follows the assertion by the uniqueness above. Similarly follow assertions (b) and (c).

(d): If $\tau = I$ this is trivial. Otherwise let $\tau = \widehat{\Omega}_\delta$. If $\psi_\tau\gamma < \beta$, follows $\delta < \beta$, $\delta \in C(\alpha,\beta)$, $\tau \in C(\alpha,\beta)$. Otherwise, if $\psi_\tau\gamma \in C^{n+1}(\alpha,\beta)\setminus C^n(\alpha,\beta)$, $\psi_\tau\gamma = \psi_\tau\gamma'$ with $\tau,\gamma' \in C^n(\alpha,\beta)$.

Lemma 10.11 *(a)* $(\rho,\pi \in R \wedge \rho < \pi \neq I) \to \rho \le \pi^-$,

(b) $\rho < I^+ \to \rho^- < \psi_\rho\alpha < \rho$,

(c) $\psi_I\alpha < \rho \neq I \to \psi_I\alpha \le \rho^-$.

Proof:
(a) trivial.
(b) 10.4 (f) and (a).
(c) If $\rho = \Omega_{\sigma+1}$, $\psi_I\alpha = \Omega_{\psi_I\alpha} < \Omega_{\sigma+1}$, then $\psi_I\alpha \le \sigma$.

Lemma 10.12 *(a)* $0 \neq \alpha \to 0 < \alpha$.

(b) If $NF_+(\alpha,\gamma)$, $NF_+(\beta,\delta)$, *then*

$$\begin{aligned}
&\alpha+\gamma < \beta+\delta \leftrightarrow \\
&\quad ((Alength(\alpha) < Alength(\beta) \wedge \alpha+\gamma \le \beta) \quad \vee (Alength(\alpha) = Alength(\beta) \wedge \alpha < \beta \vee \\
&\qquad (\alpha = \beta \wedge \gamma < \delta)) \\
&\quad \vee (Alength(\beta) < Alength(\alpha) \wedge \alpha < \beta+\delta)
\end{aligned}$$

(c) If $NF_+(\alpha,\beta)$, $\gamma \in A\setminus\{0\}$, *then*
$\alpha+\beta < \gamma \leftrightarrow \alpha < \gamma$,
$\gamma < \alpha+\beta \leftrightarrow \gamma \le \alpha$.

(d)

$$\begin{aligned}
\widehat{\phi}_\alpha\beta < \quad & \widehat{\phi}_\gamma\delta \leftrightarrow (\alpha<\gamma \wedge \beta < \widehat{\phi}_\gamma\delta) \vee (\alpha=\gamma \wedge \beta<\delta) \\
& \vee(\gamma < \alpha \wedge \widehat{\phi}_\alpha\beta \le \delta)
\end{aligned}$$

(e) If $\delta \in G$, *then:*
$\widehat{\phi}_\alpha\beta < \delta \leftrightarrow max\{\alpha,\beta\} < \delta$.
$\delta < \widehat{\phi}_\alpha\beta \leftrightarrow \delta \le max\{\alpha,\beta\}$.

(f) If $\pi,\rho \in R$, $\pi\neq\rho$, *then*
$\psi_\pi\alpha < \psi_\rho\beta \leftrightarrow$
$(I \neq \pi < \rho \neq I) \vee (I \neq \rho \wedge \pi = I \wedge \psi_\pi\alpha < \rho) \vee (\rho = I \wedge \pi < \psi_\rho\beta)$

(g) If $\pi, \rho \in R$, $\alpha \in C_\pi(\alpha)$, $\beta \in C_\rho(\beta)$, then
$\psi_\pi\alpha < \psi_\rho\beta \leftrightarrow$
$(\pi = \rho \wedge \alpha < \beta) \vee (I \neq \pi < \rho \neq I) \vee (I \neq \rho \wedge \pi = I \wedge \psi_\pi\alpha < \rho) \vee (\rho = I \wedge \pi < \psi_\rho\beta)$
Further if $\alpha \in C_\rho(\alpha)$, $\alpha < \beta$ follows $\psi_\rho(\alpha) < \psi_\rho(\beta)$.

(h) If $\rho, \pi \in R$, $\pi \neq I$, then
$\rho < \psi_\pi\beta \leftrightarrow (\rho < \pi)$
$\psi_\pi\beta < \rho \leftrightarrow (\pi \leq \rho)$

(i) $\psi_I\alpha < \widehat{\Omega}_\beta \leftrightarrow \psi_I\alpha \leq \beta$,
$\widehat{\Omega}_\beta < \psi_I\alpha \leftrightarrow \beta < \psi_I\alpha$.

(j) $\psi_I\alpha < I$.

(k) $\widehat{\Omega}_\alpha < \widehat{\Omega}_\beta \leftrightarrow \alpha < \beta$.

(l) $\widehat{\Omega}_\alpha < I \leftrightarrow \alpha < I$.
$I < \widehat{\Omega}_\alpha \leftrightarrow I \leq \alpha$.

Proof:
(a): trivial.
(b):Let $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$, $\alpha_0 \geq \cdots \geq \alpha_n$, $n = Alength(\alpha)$n,
$\beta = \omega^{\beta_0} + \cdots + \omega^{\beta_m}$, $\beta_0 \geq \cdots \geq \beta_m$, $m = Alength(\beta)$.
Then

$$\begin{aligned}\alpha + \gamma < \beta + \delta \leftrightarrow \quad & ((n < m \wedge (\forall i \leq n.\alpha_i = \beta_i) \wedge \gamma = \omega^{\beta_{n+1}}) \\ \vee \quad & (\exists i \leq \min\{n, m\}.(\forall j < i.\alpha_j = \beta_j) \wedge \alpha_i < \beta_i) \\ \vee \quad & (n < m \wedge (\forall j \leq n.\alpha_j = \beta_j) \wedge \gamma < \omega^{\beta_{n+1}}) \\ \vee \quad & (n = m \wedge (\forall j \leq n.\alpha_j = \beta_j) \wedge \gamma < \delta) \\ \vee \quad & (m < n \wedge (\forall j \leq m.\alpha_j = \beta_j) \wedge \omega^{\alpha_{m+1}} < \delta))\end{aligned}$$

If $n < m$ we have

$$\begin{aligned}\alpha + \gamma \leq \beta \leftrightarrow \quad & (\forall i \leq n.\alpha_i = \beta_i) \wedge \gamma \leq \omega^{\beta_{n+1}}) \\ \vee \quad & (\exists i \leq n.(\forall j < i.\alpha_j = \beta_j) \wedge \alpha_i < \beta_i) \\ \leftrightarrow \quad & \alpha + \gamma < \beta + \delta\end{aligned}$$

and the assertion for $Alength(\alpha) < Alength(\beta)$. The case$Alength(\alpha) = Alength(\beta)$ is obvious and the case $Alength(\beta) < Alength(\alpha)$ follows as the case $Alength(\alpha) < Alength(\beta)$.

(c) Let $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$, $\alpha_0 \geq \cdots \geq \alpha_n$. Then
$\alpha + \beta = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} + \beta < \gamma \leftrightarrow \omega^{\alpha_0} < \gamma \leftrightarrow \alpha < \gamma$, and
$\gamma < \alpha + \beta \leftrightarrow \gamma \leq \omega^{\alpha_0} \leftrightarrow \gamma \leq \alpha$.
(d) Let

$$\widetilde{\beta} := \begin{cases} \beta + 1 & \text{if } \exists n < \omega, \beta'.\beta = \beta' + n \wedge \\ & \quad (\beta' \in Cr(\alpha) \vee (\beta' = 0 \wedge \alpha \in G)) \\ \beta & \text{otherwise} \end{cases}$$

Analogously we define $\widetilde{\delta}$.
Then

$$\widehat{\phi}_\alpha \quad \beta < \widehat{\phi}_\gamma \delta$$
$$\leftrightarrow \phi_\alpha \widetilde{\beta} < \phi_\gamma \widetilde{\delta}$$
$$\leftrightarrow (\alpha < \gamma \wedge \widetilde{\beta} < \phi_\gamma \widetilde{\delta}) \vee (\alpha = \gamma \wedge \widetilde{\beta} < \widetilde{\delta}) \vee (\alpha > \gamma \wedge \phi_\alpha \widetilde{\beta} < \widetilde{\delta})$$

Further, if $\alpha = \gamma$, then $\widetilde{\beta} < \widetilde{\delta} \leftrightarrow \beta < \delta$, if $\alpha < \gamma$, then $\widetilde{\beta} < \phi_\gamma \widetilde{\delta} \leftrightarrow \beta < \phi_\gamma \widetilde{\delta}$, and if $\gamma < \alpha$, then $\phi_\alpha \widetilde{\beta} < \widetilde{\delta} \leftrightarrow \phi_\alpha \widetilde{\beta} \leq \delta$ (since if $\delta = \phi_\alpha \widetilde{\beta} \rightarrow \widetilde{\delta} = \delta + 1$), and we have the assertion.
(e) We have first $\phi_\alpha \beta < \delta = \phi_\delta 0 \leftrightarrow ((\alpha < \delta \wedge \beta < \phi_\delta 0 = \delta) \vee (\alpha = \delta \wedge \beta < 0) \vee (\delta < \alpha \wedge \phi_\alpha \beta < 0)) \leftrightarrow max\{\alpha, \beta\} < \delta$,
Therefore ($\widetilde{\beta}$ being defined as before) $\widehat{\phi}_\alpha \beta < \delta \leftrightarrow max\{\alpha, \widetilde{\beta}\} < \delta \leftrightarrow max\{\alpha, \beta\} < \delta$, and $\delta < \widehat{\phi}_\alpha \beta \leftrightarrow \neg(\widehat{\phi}_\alpha \beta < \delta) \leftrightarrow \delta \leq max\{\alpha, \beta\}$.
(f) Case $\pi, \rho \neq I$: Then $\pi^- < \psi_\pi \alpha < \pi$, $\rho^- < \psi_\rho \beta < \rho$. Therefore if $\pi < \rho$, then $\pi \leq \rho^-$, $\psi_\pi \alpha < \psi_\rho \beta$. If $\rho < \pi$, then $\rho \leq \pi^-$, $\neg(\psi_\pi \alpha < \psi_\rho \beta)$ and therefore the assertion in this case.
Case $\rho < \pi = I$: Then $\rho^- < \psi_\rho \beta < \rho$. If now $\rho \leq \psi_\pi \alpha$, follows $\psi_\rho \beta < \psi_\pi \alpha$, and if $\psi_\pi \alpha < \rho$ follows $\psi_\pi \alpha \leq \rho^- < \psi_\rho \beta$.
Case $I = \pi < \rho$: Then $\psi_\pi \alpha < \rho, \psi_\rho \beta$.
Case $\pi < \rho = I$: Then if $\pi < \psi_\rho \beta$ follows $\psi_\pi \alpha < \pi < \psi_\rho \beta$ and if $\psi_\rho \beta \leq \pi$ follows $\psi_\rho \beta \leq \pi^- < \psi_\pi \alpha$.
Case $\rho < \pi = I$: Then $\psi_\rho \beta, \rho < \psi_\pi \alpha$.

(g) If $\pi \neq \rho$ the assertion follows by (f).
Case $\pi = \rho$. If $\alpha < \beta$, $\alpha \in C_\pi(\alpha) \subset C_\pi(\beta)$ follows by 10.4 (b) $\psi_\pi \alpha < \psi_\rho \beta$, and if $\beta \leq \alpha$, $\psi_\rho \beta \leq \psi_\pi \alpha$.

(h) $\pi^- < \psi_\pi \alpha$, and $\rho < \pi \wedge \rho \leq \pi^-$ or $\pi \leq \rho$.

(i) Let $\widehat{\Omega}_\beta = \Omega_{\widetilde{\beta}}$. Then $\psi_I \alpha = \Omega_{\psi_I \alpha} < \widehat{\Omega}_\beta = \Omega_{\widetilde{\beta}} \leftrightarrow \psi_I \alpha < \widetilde{\beta} \leftrightarrow \psi_I \alpha \leq \beta$.
$\widehat{\Omega}_\beta < \psi_I \alpha \leftrightarrow \neg(\psi_I \alpha < \widehat{\Omega}_\beta) \leftrightarrow \beta < \psi_I \alpha$.

(j) trivial.
(k), (l) easy.

Lemma 10.13 *(Lemma 2.7 of [BS88]) If $\alpha < \beta$ and for all $\alpha \leq \delta < \beta$ we have $\delta \notin C_\sigma(\alpha)$, then $C_\sigma(\beta) = C_\sigma(\alpha)$ and $\psi_\sigma \beta = \psi_\sigma \alpha$.*

Proof: "$\supset$" is trivial, for "$\subset$" we prove by induction on n for $\gamma \in C^n_\sigma(\beta)$, that $\gamma \in C^n_\sigma(\alpha)$. The only difficult case is $\gamma = \psi_\tau \delta$, $\delta < \alpha$, $\tau, \delta \in C^{n-1}_\sigma(\beta)$. But in this case $\delta < \beta$, and we are done.

Lemma 10.14 *(Lemma 2.8 of [BS88]) If $\beta = min\{\xi | \alpha \leq \xi \in C_\sigma(\alpha)\}$, then $C_\sigma(\alpha) = C_\sigma(\beta)$, $\psi_\sigma \alpha = \psi \sigma \beta$, and $\beta \in C_\sigma(\beta)$.*

Proof: 10.13

Lemma 10.15 *(Corresponds to lemma [BS88]2.11.)*
Let $\tau, \gamma, \gamma_0 \in C^n_\sigma(\alpha)$, $\sigma \leq \tau \wedge \beta \leq \alpha$. Then
$\delta := min\{\xi | \gamma \leq \xi \in C_\tau(\beta)\} \in C^n_\sigma(\alpha)$,
$\delta' := min\{\xi | \gamma \leq \widehat{\phi}_{\gamma_0} \xi \in C_\tau(\beta)\} \in C^n_\sigma(\alpha)$,

Proof: Induction on n.
Case $n = 0$, $\gamma < \psi_\sigma \alpha$: If $\gamma < \psi_\tau \beta$ follows $\delta = \gamma$, $\delta' \leq \gamma < \psi_\tau \beta$.
If $\psi_\tau \beta \leq \gamma$, follows $\psi_\tau \beta \leq \gamma < \psi_\sigma \alpha \leq \sigma \leq \tau$. Since $C_\tau(\beta) \cap \tau = \psi_\tau \beta$, $\tau \in C_\tau(\beta)$ follows $\delta = \tau$, $\tau \in C^n_\sigma(\alpha)$. $\delta' < \psi_\tau \beta$ or $\delta' = \tau$, $\delta' \in C^n_\sigma(\alpha)$, similarly for δ''.

Case $n = 0$, $\gamma = 0, I$: $\delta = \gamma$, $\delta' \in \{0, I\} \subset C_\tau(\beta)$.
Case $n = n' + 1$, $\gamma = \gamma_1 + \gamma_2$, $NF_+(\gamma_1, \gamma_2)$, $\gamma_i \in C_\tau^{n'}(\beta)$: Let δ_i be chosen for γ_i. If $\gamma \le \delta_1$ follows $\delta = \delta_1$. Otherwise $\gamma_1 \le \delta_1 < \gamma_1 + \gamma_2$, $\delta_1 = \gamma_1 + \rho \in C_\tau(\beta)$, $0 < \rho < \gamma_2$ and by $NF_+(\gamma_1, \gamma_2)$ $\gamma_1 \in C_\tau(\beta)$. Therefore $\gamma_1 + \gamma_2 \le \delta \le \gamma_1 + \delta_2$, $\delta = \delta_1 + \rho$ with $\gamma_2 \le \rho \le \delta_2$, $\rho \in C_\tau(\beta)$, $\rho = \delta_2$, we see that $\delta_2 \in A$, therefore $\delta = \delta_1 + \delta_2 \in C_\tau^n(\beta)$. $\delta' = \delta_1'$, where $\delta_1' \in C_\sigma(\alpha)$ by the second IH for γ_1.
Case $n = n' + 1$, $\gamma = \widehat{\phi}_{\gamma_1}\gamma_2$, $\gamma_i \in C_\tau^{n'}(\beta)$: Let δ_i be determined for γ_i. If $\gamma \le \delta_i$ follows $\delta = \delta_i$. Let $\delta_i < \gamma$ $(i = 1, 2)$. Then $\delta_i \le \delta \le \widehat{\phi}_{\delta_1}\delta_2$, therefore $\delta \notin G$, otherwise $\delta = max\{\delta_1, \delta_2\}$.
If $\delta = \delta_3 + \delta_4$, $NF_+(\delta_3, \delta_4)$, we had $\gamma \le \delta_3 < \delta$, $\delta_3 \in C_\tau(\beta)$, a contradiction. Therefore $\delta = \widehat{\phi}_{\delta_3}\delta_4$, $\gamma \le \delta \le \widehat{\phi}_{\delta_1}\delta_2$. If $\delta_3 < \gamma_1$, we had $\gamma \le \delta_4 < \delta$, $\delta_4 \in C_\tau(\beta)$, a contradiction. Therefore $\gamma_1 \le \delta_3 \in C_\tau(\beta)$, $\delta_1 \le \delta_3$. If $\delta_1 < \delta_3$, we had $\gamma \le \delta \le \delta_2$, a contradiction, therefore $\delta_1 = \delta_3$, $\delta_4 = \delta_2' \in C_\sigma^n(\alpha)$ by the second IH for γ_2.
Second part in this Case: If $\gamma_0 < \gamma_1$ follows $\delta' = \delta$, if $\gamma_0 = \gamma_1$, $\delta' = \delta_2$, and if $\gamma_0 > \gamma_1$ chose δ_2' for γ_0, γ. If $\gamma_2 < \widehat{\phi}_{\gamma_0}\delta_2'$, $\delta = \delta_2'$, otherwise $\delta = \delta_2' + 1$.
In all cases, where $\gamma \in G$, for the second assertion we have $\delta' = 0$, if $\gamma \le \gamma_0$, $\delta' = \delta$ otherwise.
Case $\gamma = \psi_{\gamma_1}\gamma_2$, $\gamma_i \in C_\sigma^n(\alpha)$. $\gamma_2 < \alpha$. Let δ_i be chosen for γ_i.
If $\delta = \delta_3 + \delta_4$ with $NF_+(\delta_3, \delta_4)$ or $\delta = \widehat{\phi}_{\delta_3}\delta_4$, follows $\gamma \le \delta_3 < \delta$ or $\gamma \le \delta_4 < \delta$. Therefore we have $\delta \in G$. If $\gamma_1 < \delta_1$ follows $\gamma_1 \ne I$ $\delta = \delta_1$. If $\gamma_1 = \delta_1 = \delta$ or $\gamma = \delta$, we are finished, too. Therefore let $\gamma_1 = \delta_1$, $\gamma < \delta < \gamma_1$:
Subcase $\gamma_1 \ne I$: Then $\delta = \psi_{\gamma_1}\delta_3$ $\gamma_2 < \delta_3 < \beta \le \alpha$ $\delta_3^-, \delta_3 \in C_\tau(\beta)$, therefore $\delta_2 \le \delta_3$, and by minimality and since $\psi_\tau\delta_2 \le \psi_\tau\delta_3$, $\delta = \psi_\tau\delta_2 \in C_\sigma^n(\alpha)$.
Subcase $\gamma_1 = I$. If $\delta = \widehat{\Omega}_{\delta_3}$ follows $\gamma \le \delta_3 \in C_\tau(\beta)$, a contradiction, and if $\delta = \psi_{\delta_3}\delta_4$ with $\delta_3 \ne I$ follows $\gamma \le \delta_3^- < \delta$, $\delta_3 \in C_\tau(\beta)$, therefore $\delta = \psi_I\delta_4$, and as in the Subcase before follows the assertion.
Case $\gamma = I$: $\delta = I$.
Case $\gamma = 0$: $\delta = 0$, $\delta' = 0$.
Case $\gamma = \widehat{\Omega}_{\gamma_1}$: Let δ_1 be chosen for γ_1. If $\gamma \le \delta_1$, we have $\delta = \delta_1$. Otherwise follows $\delta \in G$, $\delta \ne \psi_{\delta_3}\delta_4$ with $\delta_3 \ne I$ (otherwise $\gamma \le \delta_3^-$). Therefore $\delta = I$ or $= \widehat{\Omega}_{\delta_3}$ (therefore $\delta_3 = \delta_1$) or $\delta = \psi_I\delta_3$ (but in this case $\gamma \le \widehat{\Omega}_{\delta_1} < \delta$, a contradiction).

Lemma 10.16 *Let*

$$\begin{aligned} C'^0(\alpha, \beta) &:= \beta \cup \{0, I\} \\ C'^{n+1}(\alpha, \beta) &:= C'^n(\alpha, \beta) \cup \{\widehat{\phi}_\gamma\delta, \widehat{\Omega}_\gamma | \gamma, \delta \in C'^n(\alpha, \beta)\} \\ &\quad \cup \{\gamma + \delta | \gamma, \delta \in C'^n(\alpha, \beta), NF_+(\gamma, \delta)\} \\ &\quad \cup \{\psi_\pi\xi | \pi, \xi \in C'^n(\alpha, \beta),\ \pi \in R,\ \delta < \alpha, \xi \in C_\pi(\xi)\} \end{aligned}$$

$C'(\alpha, \beta) := \bigcup_{n<\omega} C'^n(\alpha, \beta)$, $C'^n_\tau(\alpha) := C'(\alpha, \psi_\tau\alpha))$.
Then $C'(\alpha, \psi_\sigma(\alpha)) = C(\alpha, \psi_\sigma(\alpha)) = C_\sigma(\alpha)$.

Proof: $C'^n(\alpha, \beta) \subset C^n(\alpha, \beta)$, the only difficulty is, to show: $C_\sigma^n(\alpha) \subset C'^n_\sigma(\alpha)$, and here the only difficulty is the case $\gamma = \psi_\tau\beta \in C_\sigma^{n+1}(\alpha)$, $\tau, \beta \in C_\sigma^n(\alpha)$, $\beta < \alpha$. If $\tau \le \sigma$ or $\tau = I$ and $\psi_I\beta < \sigma$ follows $\gamma \le \psi_\sigma(\alpha)$, otherwise follows by 10.15 $\beta_0 := min\{\xi | \beta \preceq \xi \in C_\tau(\beta)\} \in C_\sigma^n(\alpha) \subset C'^n(\alpha)$, by 10.14 $\psi_\tau\beta = \psi_\tau\beta_0$, $\beta_0 \in C_\tau(\beta_0)$. If $\beta = \beta_0$ follows $\beta_0 < \alpha$. Otherwise $\beta \notin C_\tau(\beta_0) = C_\tau(\beta)$, if $\tau \ne I$ follows by $\sigma < \tau$ $\beta \notin C_\sigma(\beta_0)$, since $\beta \in C_\sigma(\alpha)$, $\beta_0 < \alpha$, and if $\tau = I$ we have $\sigma < \psi_I\beta$, and from $\beta \notin C_\tau(\beta_0)$ follows by $\psi_\sigma\beta_0 < \psi_\tau\beta_0$, $\beta \notin C_\sigma(\beta_0)$ and again $\beta_0 < \alpha$. Therefore $\gamma \in C_\sigma^{n+1}(\alpha)$.

Lemma 10.17 *If* $(\gamma \in C_\pi(\beta) \wedge ((\pi^- \leq \gamma \in G \wedge \pi \neq I) \vee \pi = I \leq \gamma))$, *then*

$$\exists\delta \in C_\pi(\beta).\gamma = \widehat{\Omega}_\delta) \vee$$
$$(\exists\rho, \delta \in C_\pi(\beta).\gamma = \psi_\rho\delta \wedge \delta < \beta \wedge (\pi \leq \rho \vee \rho = I) \wedge \delta \in C_\rho(\delta)).$$

Proof:
Case $\gamma < \psi_\pi\beta$: Then, since $\pi^- \leq \gamma < \psi_\pi\beta$ follows $\exists\rho \leq \beta.\psi_\pi\rho \leq \gamma < \psi_\pi(\rho+1)$, therefore $\gamma = \psi_\pi\rho$, $\rho < \beta$, and by 10.13 $\rho \in C_\pi(\rho)$.
Case $\pi \leq \gamma$: 10.16.

Lemma 10.18 *(a)* $I \neq \kappa \in R \cup I^+ \to C_\kappa(\alpha) = C'(\alpha, \kappa^- + 1)$.

(b) $C_{\Omega_1}(I^+) = C'(I^+, 0)$

Proof:
(a) "$\supset$" is obvious. For "$\subset$" we prove by induction on α:
$\rho \in \psi_\kappa\alpha \to \rho \in C'(\alpha, \kappa^- + 1)$ ($\Rightarrow \psi_\kappa\alpha \subset C'(\alpha, \kappa^- + 1)$ and further $C'(\alpha, \kappa^- + 1) = C'(\alpha, \psi_\kappa\alpha) = C_\kappa(\alpha)$).
If $\rho \leq \kappa^-$, this is obvious, and if $\rho = \rho_1 + \rho_2$, $NF_+(\rho_1, \rho_2)$, or $\rho = \widehat{\phi}_{\rho_1}\rho_2$ or $\rho = \widehat{\Omega}_{\rho_1}$ this follows by IH. Otherwise follows $\exists\delta.\delta \in C_\kappa(\delta) \wedge \delta < \alpha \wedge \rho = \psi_\kappa\delta$. Then $\delta \in C_\kappa(\delta) = C'(\delta, \kappa^- + 1) \subset C'(\alpha, \kappa^- + 1)$ by IH, $\psi_\kappa\delta \in C'(\alpha, \kappa^- + 1)$.
(b): $C_{\Omega_1}(I^+) = C'(I^+, \omega + 1) = C'(I^+, 0)$.

Definition 10.19 *Definition of* $G_\pi(\alpha)$ *for* $\alpha \in C_{\Omega_1}(I^+) = C'(I^+, 0)$ *by recursion on the minimal* n *such that* $\alpha \in C'^n(I^+, 0)$.

(G1) $G_\pi 0 := \emptyset$.
(G2) $NF_+(\alpha, \beta) \to G_\pi(\alpha + \beta) := G_\pi\alpha \cup G_\pi\beta$.
(G3) *If* $\rho\ \eta\ R$, $\beta \in G_\rho(\beta)$, *then*

$$G_\pi\psi_\rho\beta := \begin{cases} \{\beta\} \cup G_\pi\rho \cup G_\pi\beta, & \text{if } \pi \leq \rho \neq I \vee \\ & \rho = I \wedge (\pi \leq \psi_I\beta \vee \pi = I), \\ G_\pi\rho & \text{if } \rho < \pi = I \\ \emptyset, & \text{if } \rho < \pi \neq I \text{ or} \\ & \rho = I \wedge \psi_I\beta < \pi < I. \end{cases}$$

(G4) $G_\pi(\widehat{\Omega}_\alpha) := G_\pi\alpha$.
(G1) $G_\pi I := \emptyset$.

Lemma 10.20 *If* $\alpha \in C_{\Omega_1}(I^+)$, *then* $\alpha \in C_\pi(\beta) \leftrightarrow G_\pi(\alpha) < \beta$.

Proof:
Induction on n, such that $\alpha \in C'(I^+, 0)$.
If $\alpha = \gamma + \delta$ and $NF_+(\gamma, \delta)$ or $\alpha = \widehat{\phi}_\gamma\delta, \widehat{\Omega}_\gamma, I, 0$ the assertion follows by IH or immediately.
Let $\alpha = \psi_\rho\xi$, $\xi \in C_\rho(\xi)$, $\xi, \rho \in C'(I^+, 0)$.
If $\pi = \rho$ follows $\alpha \in C_\pi(\beta) \leftrightarrow \alpha < \psi_\pi\beta \leftrightarrow \xi \in C_\pi(\beta) \wedge \xi < \beta$ (using $\xi \in C_\pi(\xi)$) $\leftrightarrow \xi, \pi \in C_\pi(\beta) \wedge \xi < \beta \leftrightarrow G_\pi(\alpha) < \beta$.
If $\rho < \pi \neq I$ follows $G_\pi(\alpha) = \emptyset$, $\alpha \in C_\pi(\beta)$.
If $\pi < \rho \neq I$ follows $\alpha \in C_\pi(\beta) \leftrightarrow \rho, \xi \in C_\pi(\beta) \wedge \xi < \beta$.
If $\pi < \rho = I$ follows, if $\psi_\rho\xi < \pi$ $\psi_\rho\xi \in C_\pi(\beta)$, $G_\pi(\alpha) = \emptyset$, and if $\pi \leq \psi_\rho\xi$, $\psi_\rho\xi \in C_\pi(\alpha) \leftrightarrow \rho, \xi \in C_\pi(\beta) \wedge \xi < \beta \leftrightarrow G_\pi(\alpha) < \beta$.
If $\rho < \pi = I$ follows $\alpha \in C_\pi(\beta) \leftrightarrow \psi_\rho\xi < \psi_\pi\beta \leftrightarrow \rho < \psi_\pi\beta \leftrightarrow \rho \in C_\pi(\beta) \leftrightarrow G_\pi\rho < \beta \leftrightarrow G_\pi(\alpha) < \beta$.

Lemma 10.21 *Let* $a, u, c \in OT$. *Then follows:*

(a) $o(a) \in C_{\Omega_1}(I^+)$.

(b) $a \in G \to o(a) \in G$, *similarly for* Lim, Suc, A, R, Fi.

(c) $G_o(u)(o(a)) = \{o(x) \mid x \in G_u a\}$.

(d) $a \prec d \Rightarrow o(a) < o(d)$.

Proof by induction on $length(a) + length(u)$, simultaneously for (a) - (d):
1. $a = D_b c$: Then $G_b c \prec c$ and $b, c \in OT$.
(a) By IH $o(b), o(c) \in C_{\Omega_1}(I^+)$ und $G_{o(b)} o(c)$
$= \{o(x) \mid x \in G_b c\} < o(c)$. By lemmata 10.20 follows
$o(b) \in I^+ \cap C_{o(b)}(o(c))$ and therefore $o(a) = \psi_{o(b)} o(a) \in C_{\Omega_1}(I^+)$.
(b) trivial.
(c) Immediate by IH and definition of $G_u a$.
(d) follows by side induction on $length(d)$ using 10.12. The only difficulty is the case $d = D_e f$. If $e \neq b$ we use 10.12 (f), and if $b = e$ we have $a \prec d \leftrightarrow c \prec f \rightarrow o(c) < o(f) \rightarrow o(a) \prec o(d)$ by 10.12 (g).
2. All other cases follow immediately, using in (c) again side induction on $length(d)$ and 10.12.

Lemma 10.22 *For all* $\alpha \in C'^n(I^+, 0)$ *exists* $a \in OT$ $\alpha = o(a)$.

Proof:
If $\alpha = 0, I$ this is immediate, and if $\alpha = \gamma + \delta$ with $NF_+(\gamma, \delta)$ or $\alpha = \widehat{\phi}_\gamma \delta, \widehat{\Omega}_\gamma$ this follows by IH for γ, δ and if $\alpha = \psi_\rho \delta$, $\delta \in C_\rho(\delta)$, that is $G_\rho(\delta) < \delta$ follows $\rho = o(r)$ for some $r \in R$, $\delta = o(d)$ for some $d \in OT$, $G_r(d) < d$ by 10.21, $\alpha = o(D_r d)$ with $D_r d \in OT$.

Proof of lemma 10.7: (a) is proven. Further $\{o(x) | x \prec \widehat{\Omega}_0 \wedge x \in OT\} = C_{\Omega_1}(I^+) \cap \Omega_1 = \psi_{\Omega_1} I^+$, and $o(\cdot)$ is an order preserving map $\{x | x \prec \widehat{\Omega}_0 \wedge x \in OT\} \rightarrow \psi_{\Omega_1} I^+$,
and for $a \prec \widehat{\Omega}_1$, $\{o(x) | x \prec a \wedge x \in OT\} = C_{\Omega_1}(I^+) \cap o(a) = o(a)$, again $o(\cdot)$ is an order preserving isomorphism.

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