

# **Some new links between combinatorial theory and the satisfiability problem**

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# Overview

- I On a conjecture of Endre Boros
- II Conflict matrices (aka conflict graphs)
- III Hermitian Defect (and the deficiency)

## [Some notations]

Let  $\mathcal{CL}$  be the set of clauses (finite, complement-free sets of literals).

$\mathcal{MCLS}$  denotes the set of finite multi-clause-sets, these are maps  $F : \mathcal{CL} \rightarrow \mathbb{N}_0$  with finite support.

The **deficiency** of  $F \in \mathcal{MCLS}$  is

$$\delta(F) := c(F) - n(F),$$

where  $c(F) := \sum_{C \in \mathcal{CL}} F(C)$  is the number of clause occurrences in  $F$ , and  $n(F) := |\text{var}(F)|$  is the number of variables in  $F$ . The **maximal deficiency** is  $\delta^*(F) := \max_{F' \leq F} \delta(F')$ .

For a partial assignment  $\varphi$  and a multi-clause-set  $F$  by  $\varphi * F$  the result of substituting  $\varphi$  into  $F$  is denoted:

$$(\varphi * F)(C) = \sum_{\substack{C' \in \mathcal{CL} \\ \varphi * \{C'\} = \{C\}}} F(C').$$

## Some notations

The **deficiency** of a multi-clause-set  $F$  is

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is the number of variables in  $F$ .

The **maximal deficiency** is

$$\delta^*(F) := \max_{F' \leq F} \delta(F')$$

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# I On a conjecture of Endre Boros

A **hitting clause-set** is a multi-clause-set  $F$  such that for all clauses  $C, D \in F$  (i.e.,  $F(C), F(D) \neq 0$ ) we have  $C \cap \overline{D} \neq \emptyset$ .

Every two clauses clash in at least one literal.

A **1-uniform hitting clause-set** fulfils  $|C \cap \overline{D}| = 1$  for all clauses  $C, D \in F$ .

Every two clauses clash in exactly one literal.

At the SAT'98 workshop in Paderborn Endre Boros conjectured, that for every 1-uniform hitting clause-set  $F$  with  $\delta(F) = 1$ ,  $n(F) > 0$  there is a literal  $x$  such that  $\#_x(F) = 1$  (where  $\#_x(F) = \sum_{x \in C \in \mathcal{C}\mathcal{L}} F(C)$ ).

Xishun and Kleine Büning proved, that

- $F \in \mathcal{UHIT}_1$ ,  $\delta(F) \geq 2 \Rightarrow F \in \mathcal{SAT}$
- If the conjecture of Endre Boros is true, then for  $F \in \mathcal{UHIT}_1$  we have  $\delta(F) \leq 1$ .

# A counter example, and the Theorem of Witsenhausen

A smallest counter example to the conjecture of Endre Boros has the clause-variable matrix

$$\begin{pmatrix} + & & + & & & & + \\ + & & - & + & & & \\ - & - & & & & + & + \\ - & - & & & - & - & \\ & + & - & - & - & & \\ & & & - & + & & - & - \\ - & & & & + & - & & + \\ - & & & + & & + & - & \\ & + & + & & & & + & - \end{pmatrix} .$$

[*Starfree biclique decompositions of complete graphs*; Allen J. Schwenk and Ping Zhang, 1998]

And by the “Theorem of Witsenhausen”, that every addressing of the complete graph  $K_m$  needs at least  $m - 1$  positions, in fact every 1-uniform hitting clause-set has deficiency at most 1.

## **[On the addressing problem for loop switching (For funding agencies: SAT and the Internet(!))]**

In the early seventies, Pierce proposed a scheme for transmission of information in a network by establishing “loops” (back-and-forth channels for packages) and “loop switches” .

In [*On the addressing problem for loop switching*; R.L. Graham and H.O. Pollak, Bell System Technical Journal, 1971]

a realisation was proposed as follows:

Every node  $w$  of the network gets an “address”  $A(w)$ . The “distance”  $d(A(w), A(w'))$  equals the distance of  $w$  and  $w'$  in the network (the length of a shortest path in the graph).

Now a package we want to sent from node  $u$  and  $v$  can find its way “locally” through the network by switching at each node  $w$  to a node  $w'$  such that  $d(A(w'), A(v)) = d(A(w), A(v)) - 1$ .

## [Which addressing schemes?!

What codes do we use?!

Hamming codes?! We need a distance preserving embedding for each graph in some Hamming code (i.e., addresses are words over  $\{0, 1\}$ , the distance is the number of positions with different values).

For example the cycle with three nodes does not have such an embedding.

(Every embeddable graph must be bipartite, since the Hypercube  $\{0, 1\}^n$  as a graph is bipartite (use two colour classes “even” and “odd” according to the parity of the sum of the coordinates).)

So Graham and Pollak introduced a “don’t care” symbol  $*$  (thus the code is a ternary code now), where positions, where at least one of the addresses has a  $*$ , are ignored for the Hamming distance.

Now an addressing exists for every graph.

The problem is to minimise the number of positions for the code.

## [“Addresses” are just clauses]

Consider a set of addresses, that is, a set of elements of  $\{0, 1, *\}^n$ . Take variables  $v_1, \dots, v_n$ , interpret

- “0” as “positive”
- “1” as “negative”
- “\*” as “not there”

In this way every address corresponds to a clause, and the distance between addresses becomes the number of clashing literals!

Considering the distance matrix  $D$  of a graph, Graham and Pollak proved by a nice application of Sylvester’s Law of Inertia, that at least

$$\max(i_+(D), i_-(D))$$

many positions are needed, where  $i_{\pm}(D)$  is the number of positive resp. negative eigenvalues of  $D$  ( $D$  is a symmetric real matrix).

# The addressing problem for graphs revisited

Given a connected graph  $G$ , the **addressing problem** of Graham and Pollak is (naturally) reinterpreted as the problem of

finding a multi-clause-set  $F$  such that the *conflict matrix* of  $F$  is the distance matrix  $D(G)$  of  $G$ , and such that  $n(F)$  is minimal.

Example:  $G$  is  $\bullet \text{---} \bullet \text{---} \bullet$

$$D(G) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

An exact “addressing”  $F$  is  $\{\{a, b\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}\}\}$ .

**Theorem of Graham and Pollak**  $n(F) \geq \max(i_+(D(G)), i_-(D(G)))$ , where  $i_{\pm}(D)$  is the number of positive resp. negative eigenvalues of  $D$  ( $D$  is a symmetric real matrix).

## Addressing the complete graph

Consider  $K_m$ , the complete graph with  $m$  nodes. Using the saturated minimally unsatisfiable Horn formula

$$\left\{ \{v_1\}, \{\overline{v_1}, v_2\}, \{\overline{v_1}, \overline{v_2}, v_3\}, \dots, \{\overline{v_1}, \dots, \overline{v_{m-2}}, v_{m-1}\}, \{\overline{v_1}, \dots, \overline{v_{m-1}}\} \right\}$$

we obtain an addressing of  $K_m$  with  $m - 1$  variables.

On the other hand, the distance matrix of  $K_m$  is  $J_m - I_m$ ,  $J_m$  the all-one square matrix of order  $m$ ,  $I_m$  the diagonal matrix of order  $m$ .

$J_m - I_m$  has eigenvalue  $-1$  with multiplicity  $m - 1$ , since

$$(J_m - I_m)(x) = (-1) \cdot x \Leftrightarrow J_m \cdot x = 0,$$

where  $J_m$  has rank 1.

Thus by the theorem of Graham and Pollak every “exact” addressing of the complete graph  $K_m$  has  $m - 1$  positions.

In other words, the deficiency of a 1-uniform hitting clause-set is at most one:

$$F \in \mathcal{UHIT}_1 \Rightarrow \delta(F) \leq 1.$$

The program now is to make a theory out of this, using the language of (multi-)clause-sets, and ideas from that field.

## II Symmetric conflict matrices

The **symmetric conflict matrix**  $C_s(F)$  of  $F \in \mathcal{MCLS}$  is the square matrix of order  $c(F)$ , where the entry at position  $(i, j)$  is the number of clashes between clauses number  $i$  and  $j$  of  $F$  (using some ordering of the clause-occurrences of  $F$ ).

Symmetric conflict matrices are symmetric non-negative integer matrices with zero diagonal. (Every such matrix is the symmetric conflict matrix of some multi-clause-set.)

The **symmetric conflict number**  $n_s(A)$  of a symmetric conflict matrix  $A$  is the minimal  $n(F)$  for  $F \in \mathcal{MCLS}$  with  $C_s(F) = A$ .

A multi-clause-set  $F$  is called **exact** if  $n(F) = n_s(C_s(F))$ .

If  $A$  is the distance matrix of some graph  $G$ , then  $n_s(F)$  is the minimal length of an addressing of  $G$ . The optimal addressings of  $G$  correspond to exact multi-clause-sets with symmetric conflict matrix  $A$ .

## Conflict graphs and biclique partitions

The **conflict graph** of  $F \in \mathcal{MCLS}$  is the graph with adjacency matrix  $C_s(F)$ , i.e., the nodes are the clause occurrences of  $F$ , and if clauses  $C, D$  clash in  $p$  literals, then we have  $p$  parallel edges between  $C$  and  $D$  in the conflict graph.

The edges of the conflict graph are labelled with the variables responsible for the corresponding conflict.

The subgraph  $B_v$  induced by the edges labelled with some variable  $v \in \text{var}(F)$  is a complete bipartite graph.

Different  $B_v$  have no edge in common, and every edge of the conflict graph of  $F$  is contained in some  $B_v$ .

Thus every multi-clause-set yields a **biclique partition** of its conflict graph. Up to pure literals, the multi-clause-set can be recovered from this biclique partition.

Every biclique partition of a graph  $G$  with adjacency matrix  $A$  (a non-negative integral matrix with zero diagonal) comes from some multi-clause-set.

Thus the **biclique partition number** of a graph  $G$  is the symmetric conflict number of the adjacency matrix of  $G$ .

A “star” (or “claw”) is a biclique, where one side has only one node. This corresponds to variables occurring in one sign only once.

Thus any exact star-free biclique cover of the complete graph  $K_m$  is a counterexample to the conjecture of Endre Boros. (And such biclique covers of  $K_m$  exist iff  $m \geq 9$ .)

## Principal submatrices

Consider  $F' \leq F$ . Then  $C_s(F')$  is a principal submatrix of  $C_s(F)$  (obtained from  $C_s(F)$  by removing rows and columns with the same indices).

Since  $\varphi * F$  results from  $F$  by first removing all clauses satisfied by  $\varphi$ , and then eliminating all literal occurrences falsified by  $\varphi$  from the remaining clauses, and in these remaining clauses those literal occurrences have become pure literals, also  $C_s(\varphi * F)$  is a principal submatrix of  $F$ .

These simple observations seem to me of fundamental importance.

A lot is known on the connections between matrices and principal submatrices; most important here are **eigenvalue techniques**.

Using “eigenvalue relaxations” we can thus connect properties of  $F$  and  $\varphi * F$  !

## [Formula classes by matrix classes]

Consider a class  $\mathcal{C}$  of matrices stable under principal-submatrix formation. Then the class  $\mathcal{MCLS}(\mathcal{C})$  of  $F \in \mathcal{MCLS}$  with  $C_s(F) \in \mathcal{C}$  is stable under sub-multi-clause-set formation and application of partial assignments. Examples are:

1.  $\mathcal{HIT}$ , the class of **hitting clause-sets**  $F$ , characterised by  $C_s(F) \geq 2(J_{\mathcal{C}(F)} - I_{\mathcal{C}(F)})$ ;
2.  $\mathcal{UHIT}_k$ , the class of  **$k$ -uniform hitting clause-sets**  $F$  for  $k \geq 1$ , characterised by  $C_s(F) = k(J_{\mathcal{C}(F)} - I_{\mathcal{C}(F)})$ ;
3.  $\mathcal{UHIT} = \bigcup_k \mathcal{UHIT}_k$
4. the class of “resolution-free” multi-clause-sets, where  $C_s(F)$  has no entry equal to 1;
5.  $\mathcal{MCLS}_h(k) := \{F \in \mathcal{MCLS} : \delta_h(F) \leq k\} \subseteq \mathcal{MCLS}(k) = \{F \in \mathcal{MCLS} : \delta^*(F) \leq k\}$ , where  $\delta_h(F)$  is the **hermitian defect**.

## [The asymmetric conflict number]

The addressing problem and the biclique partition problem has been considered for directed graphs as well (considering only directed paths resp. demanding, that in a biclique all arrows are directed “left” to “right”).

For  $F \in \mathcal{MCLS}$  let  $C_a(F)$  be the square matrix of order  $c(F)$ , such that at position  $(i, j)$  we have the number of **positive** literals  $v$  in  $C_i$  with  $\bar{v} \in C_j$ .

We have  $C_s(F) = C_a(F) + C_a(F)^t$ .

A **conflict matrix** is a non-negative square integral matrix with zero diagonal. Conflict matrices are exactly the asymmetric conflict matrices of multi-clause-sets.

The **asymmetric conflict number**  $n_a(A)$  of a conflict matrix  $A$  is the minimal  $n(F)$  for  $F \in \mathcal{MCLS}$  with  $C_a(F) = A$ .

## Fundamental characterisations

Consider a conflict matrix  $A$ .

- 1(a) The asymmetric conflict number  $n_a(A)$  is the minimal  $q \in \mathbb{N}_0$ , such that there are  $\{0, 1\}$ -matrices  $B_1, \dots, B_q$  of rank 1 with  $A = B_1 + \dots + B_q$ .
  - (b)  $n_a(A) \geq \text{rank}(A)$ .
  - (c)  $n_a(A)$  is the minimal  $k \in \mathbb{N}_0$  such that there are  $\{0, 1\}$  matrices  $X, Y$  with  $X \cdot Y = A$ , where  $X$  has  $k$  columns.
2. Assume  $A$  is symmetric.
    - (a)  $n_s(A)$  is the minimal  $q \in \mathbb{N}_0$ , such that there are symmetric  $\{0, 1\}$ -matrices  $B_1, \dots, B_q$  of rank 2 with  $A = B_1 + \dots + B_q$ .
    - (b)  $n_s(A) \geq \frac{1}{2} \text{rank}(A)$ .
    - (c)  $n_s(A)$  is the minimum of  $n_a(B)$  for conflict matrices  $B$  with  $B + B^t = A$ .

### III The hermitian defect

For a conflict matrix  $A$ ,  $n_s(A)$  is the minimal  $q$  such that there are  $\{0, 1\}$ -matrices  $X, Y$  with  $A = XY + Y^t X^t$ , where  $X$  has  $q$  columns and  $Y$  has  $q$  rows.

For any symmetric real matrix  $A$  let  $i_+(A)$  be the number of positive eigenvalues of  $A$ , and let  $i_-(A)$  be the number of negative eigenvalues of  $A$ .

Following [*Biclique decompositions and Hermitian rank*; David A. Gregory and Valerie L. Watts and Bryan L. Shader, 1999], the *hermitian rank*  $h(A)$  of  $A$  now is defined as

$$h(A) := \max(i_+(A), i_-(A)).$$

It is  $h(A)$  the minimal  $q$  such that there are real matrices  $X, Y$  with  $A = XY + Y^t X^t$ , where  $X$  has  $q$  columns and  $Y$  has  $q$  rows.

A conflict matrix:  $n_s(A) \leq h(A)$ .

## The hermitian defect

For a real symmetric matrix  $A$  of order  $m$  let  $\delta_h(\mathbf{A}) := m - h(A)$  be the **hermitian defect**. And for  $F \in \mathcal{MCLS}$  let  $\delta_h(F) := \delta_h(C_s(F))$ .

Graham-Pollak reformulated:  $\delta(F) \leq \delta_h(F)$ .

By Cauchy's interlacing inequalities:

For every  $F' \leq F$  and every partial assignment  $\varphi$  we have

$$\delta(F') \leq \delta_h(F), \quad \delta(\varphi * F) \leq \delta_h(F),$$

and thus  $\delta^*(F) \leq \delta_h(F)$ .

In fact we have the **Theorem**:

$$F' \not\leq F \Rightarrow \delta^*(F') < \delta_h(F).$$

## Eigensharp multi-clause-sets

$F \in \mathcal{MCLS}$  is called **eigensharp** if  $\delta_h(F) = \delta(F)$ .

$F$  eigensharp  $\Rightarrow F$  exact.

While exactness of  $F$  is co-NP complete, eigensharpness of  $F$  is decidable in polynomial time.

**Theorem:** Every eigensharp multi-clause-set is matching lean, where  $F$  being matching lean is equivalent to each of the following conditions:

1.  $F$  has no non-trivial matching autarky.
2. For all  $F' \leq F$  we have  $\delta(F') < \delta(F)$ .
3. The transversal matroid, which is naturally associated with the bipartite clause-variable graph of  $F$  (where the nodes on the one side are the clauses (with multiplicities), while on the other side we have the variables), is cyclic.

4. For every  $\emptyset \neq V \subseteq \text{var}(F)$  and  $F_V \in \mathcal{MCLS}$  defined by  $F_V(C) := F(C)$  for  $C \in \mathcal{CL}$  with  $\text{var}(C) \cap V \neq \emptyset$ , while in case of  $\text{var}(C) \cap V = \emptyset$  we set  $F_V(C) := 0$ , we have  $c(F_V) \geq |V| + 1$ .
5.  $F = \top$ , or we have  $c(F) \geq n(F) + 1$ , and every  $n(F) \times n(F)$ -submatrix of  $M(F)^t$  is fully indecomposable.

**Conjecture:** Every eigensharp multi-clause-set is linearly lean.

**Problem:** Find an example for an unsatisfiable eigensharp multi-clause-set which is not lean.

# Generalising the squashed cube conjecture

1984 Peter Winkler proved (now expressed in our language):

Consider an *exact*  $F \in \mathcal{MCLS}$  such that  $C_s(F)$  is the distance matrix of some connected graph. Then  $\delta(F) \geq 1$  holds.

**Conjecture:** If the resolution graph of  $F \in \mathcal{MCLS}$  is connected,  $F$  is exact and the entries of  $C_s(F)$  are less or equal than  $n(F) - 1$ , then  $F$  is matching lean.

Here the **resolution graph** of  $F$  has the clauses of  $F$  as nodes, and an edge connecting two nodes if these clauses clash in *exactly* one literal. (Thus the resolution graph is a subgraph of the conflict graph.)

# Multi-clause-sets with minimal hermitian defect

We know  $UHIT \subset MCLS_h(1)$ .

**Theorem:** For  $F \in MCLS_h(1)$  we have

$F$  is matching lean iff  $F$  is eigensharp iff  $\delta(F) = 1$ ;

and  $F$  is matching satisfiable (i.e.,  $\delta^*(F) = 0$ ) iff  $\delta(F) \leq 0$ .

**Theorem:** The unsatisfiable multi-clause-sets with hermitian defect one are exactly the saturated minimally unsatisfiable clause-sets with defect one.